PROBABILISTIC MODELLING OF EXTREME WAVE CRESTS: A NOISY WEIBULL MODEL

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Abstract

We seek to develop a probabilistic model of extreme wave crests using three separate modelling steps:

(1) We suggest and test random models of an arbitrary wave height, H, selected from a prescribed model test seastate (with given wave spectrum, characterized for example by significant wave height, H_S, and peak spectral period, T_P).

(2) We suggest and test deterministic models of the average crest height given a specified total peak-to-trough height, H.

(3) Finally, we suggest and test "noise" models of the variability of individual crests about their average value, as given in step (2).

We pursue this three-fold procedure for several reasons. First, its preliminary focus on wave height permits our use and evaluation of relatively well established probabilistic models, such as the Rayleigh model and various Weibull generalizations. Then, its secondary focus on the deterministic, mean trend in crest height given wave height -- in other words, with the average degree of wave asymmetry -- provides a useful comparison point with physical deterministic theory (e.g. Stokes theory). Finally, the noise term in step (3) requires explicit recognition of random wave behavior. On the assumption, however, that this noise term is "small" compared with the underlying wave height variability in step (1), analytical approximations can be derived that apply a small perturbation to the Rayleigh/Weibull wave height model. The result is hence referred to here as a "noisy" Weibull model of wave crests.

Background

Probabilistic models of extreme wave crest levels -- the distance between the mean water level and the maximum wave level -- are useful to predict extreme loads in severe storms (e.g., with large values of the significant wave height, H_S, and associated peak spectral period T_P). Unfortunately, wave crest statistics have been generally less studied than have the total peak-to-trough wave heights H. Common crest models to date are based on either non-linear wave theory (e.g., Tayfun, 1980; Huang *et al.*, 1983) or empirical data (e.g., Haring *et al.,* 1976; Forestall, 1978). Potential drawbacks here are that the theoretical models may be insufficiently nonlinear (Jha and Winterstein, 2000), while the empirical models may be too location-specific (e.g., in terms of water depth and degree of short-crestedness).

When waves remain below the deck of fixed structures, it can be debated whether the crest height, C, or total wave height, H, is a better indicator of the resulting base shear force and overturning moment. It is clear, however, that the crest height is the dominant parameter in assessing the likelihood of wave-in-deck impact and its resulting severity. Moreover, the crest height C generally depends critically on the degree of non-linearity displayed by the wave, while the resulting peak-to-trough height (which we may predict quite accurately) gives only a weak indicator of these nonlinear trends. To first approximation, the main effect of non-linearity is to raise both the peak and trough by a common amount, thereby showing no effect in the resulting wave height.

Data

Our ultimate goal here is to derive and test a practical and simple probabilistic model of extreme wave crests -- yielding, for example, a closed-form result for the crest height level as a function of the desired return period. Our testing phase is accomplished with the help of a singularly useful data set: model test experiments which recorded 6 statistically independent realizations of the same 3+ hour seastate, resulting in roughly 20 hours of waves in that seastate. A similar set of 20-hour realizations were obtained in three different seastates, with H_S values ranging from 14.5-16.0m. (The original goal of these tests was to focus on the ringing response of a TLP, whose rather rare transient response events -- roughly 1 per hour -- motivated the test length.) Properties of the data sets are summarized in Table 1. In this table H_S = significant wave height, T_P = peak spectral period, L_P = peak spectral length, and S_P = steepness.

	Hs	Tp	Lp=(g/2π)Tp^2	Sp=Hs/Lp
	(m)	(sec)	(m)	
ULS1	15.5	17.8	495	0.031
ULS2	15.0	16.7	435	0.034
ULS3	14.5	16.0	400	0.036

Table 1

Probabilistic Modelling Steps

As noted above, we proceed here in three modelling steps, considering in turn the total wave height *H*, the mean crest, $\overline{C}(H)$, given the total wave height *H*, and the standard deviation, $\sigma_{C|H}$, of the crest given the total wave height *H*.

1. Modelling Total Wave Heights

We first seek accurate probabilistic models of *H*, the total peak-to-trough wave height, in an arbitrary cycle of a stationary seastate. In particular, it is common to fit some form of a Weibull distribution to *H*. The general Weibull distribution form is

$$P[H_{norm} > h_{norm}] = G_{H_{norm}}(h_{norm}) = \exp\left[-\left(\frac{h_{norm}}{\alpha_H}\right)^{\beta_H}\right] \quad (1)$$

in which $H_{norm} = H/\sigma$, the wave height normalized by the standard deviation, σ , of the underlying random process. (We normalize here by the theoretical value, $\sigma = H_S/4$, which is found to vary negligibly from the value found for the wave tank histories.)

Linear random vibration theory would suggest that irrespective of H_S and T_P, the normalized height H_{norm} should follow a Rayleigh distribution, a special case of Eq. 1 in which $\beta_H = 2$ and $\alpha_H = 2\sqrt{2} = 2.82$. Figure 1 suggests that data from the three seastates, when normalized by their respective standard deviations, do appear to yield similar distributions of H_{norm}. Further, the Rayleigh model appears to yield a rather accurate prediction in all cases. (The figure also shows a slightly modified Weibull distribution, suggested by Forestall (1978), with parameter values $\beta_H = 2.126$ and $\alpha_H = 2.72$). Figure 2 shows these same results on Weibull scale, where both Rayleigh and Forestall models become straight lines. We retain the Rayleigh here, in view of both its simplicity and superior fit in this case. One may question why actual waves -- which are neither linear nor ideally narrow-band -- should follow the Rayleigh model, which assumes both properties. This may be a result of offsetting effects: nonlinear effects, which tend to enhance wave heights, may be offset here by bandwidth effects, which tend to reduce these heights.

2. Modelling Mean Crests Heights

We next consider the expected mean crest, $E[C|H]$, given the total wave height *H*. For notational simplicity, we denote this by $\overline{C}(H)$. As before, in an effort to reduce the sensitivity of results to σ, we seek to relate normalized versions of *C* and *H*; i.e., the unitless quantities $C_{norm} = C/\sigma$ and $H_{norm} = H/\sigma$.

Figure 3 shows that when the data are normalized, mean crest results across the three seastates indeed show similar behavior. Numerical results shown here reflect mean crest levels found when data are sorted into 30 equally spaced *H* bins. From these 30 binned observations of $\overline{C}_{norm}$, the following power law has been fit (through linear regression on log scale):

$$\overline{C}_{norm}(H_{norm}) = k_1 (H_{norm})^{k_2}\,; \quad k_1 = 0.53,\ k_2 = 1.03 \tag{2}$$

Separate regressions for the three seastates give k_1=0.53, and k_2=1.02-1.03. Clearly, the asymmetry among the (normalized) crest data is primarily modelled by the leading coefficient, i.e., mean crests that are 53% of the total peak-to-trough height. We have also found very similar trends from deterministic nonlinear wave theory (e.g., second-order Stokes). The power-law form in Eq. 2 has the virtue of convenience: together with the Weibull model for H_{norm} in Eq. 1, it implies a corresponding Weibull model for $\overline{C}_{norm}$ with parameters

$$\beta_{\overline{C}} = \beta_H / k_2\,; \quad \alpha_{\overline{C}} = k_1 \alpha_H^{k_2} \tag{3}$$

3. Modelling Crest Height Variability

Figure 4 shows the corresponding standard deviations of the normalized crests C_{norm}, based on the same binning with respect to the wave height as in Figure 3. Although these standard deviations show somewhat more variability than the corresponding means, reasonable agreement is found among the three seastates. Noting that our interest lies in accurate models for large waves (large *C*, *H* values), we adopt here the constant value

$$\sigma_{C_{norm}|H_{norm}} = \sigma_{\varepsilon} = 0.4 \tag{4}$$

Physically, this states that if the wave height is known, the conditional standard deviation of its associated crest C is 40% of σ, the marginal standard deviation of the process. (The accuracy of all foregoing assumptions will be tested below, in the practical context of predicting the marginal distribution of crest heights.)

Resulting Crest Height Distribution

From the preceding three steps, we may now estimate the desired quantity of interest; namely, the implied probability distribution of crest heights. The result will involve five parameters: the parameters α_H, β_H of the Weibull wave height distribution (Eq. 1), the parameters k_1 and k_2 that describe the mean crest $\overline{C}(H)$ as a function of H, and the standard deviation $\sigma_{C_{norm}} = \sigma_{\varepsilon}$ that describes a mean-zero error term, ε, to be added to the mean trend $\overline{C}_{norm}(H)$:

$$C_{norm} = \overline{C}_{norm} + \varepsilon \tag{5}$$

The resulting distribution of (normalized) crest heights can be estimated as follows:

$$P[C_{norm} > c_{norm}] = G_{C_{norm}}(c_{norm}) = \int_0^{\infty} G_{\overline{C}_{norm}}(c_{norm} - \varepsilon) f(\varepsilon) d\varepsilon \tag{6}$$

$$= E_{\varepsilon}\left[G_{\overline{C}_{norm}}(c_{norm} - \varepsilon)\right] \tag{7}$$

$$\approx G_{\overline{C}_{norm}}(c_{norm}) + \frac{\sigma_{\varepsilon}^2}{2} G''_{\overline{C}_{norm}}(c_{norm}) \tag{8}$$

The approximation in Eq. 7 results from a second-order Taylor approximation, which provides approximate results independent of the specific probability distribution form adopted for ε. Substituting the Weibull distribution form for $G_{C_{notm}}$, we get

$$P[C_{norm} > c_{norm}] \approx \exp\left[-\left(\frac{c_{norm}}{\alpha_{\overline{C}}}\right)^{\beta_{\overline{C}}}\right] \times \left\{1 + \frac{1}{2}\left(\frac{\beta_{\overline{C}}\sigma_{\varepsilon}}{\alpha_{\overline{C}}}\right)^2 \left(\frac{c_{norm}}{\alpha_{\overline{C}}}\right)^{2\beta_{\overline{C}}-2}\right\} \tag{9}$$

where $\alpha_{\overline{C}} = k_1 \alpha_H^{k_2}$ and $\beta_{\overline{C}} = \beta_H / k_2$.

Figures 5 and 6 show -- on different scales -- results predicted from the "noisy" Weibull model in Eq. 9, together with a deterministic Weibull model (in which the noise term is neglected, and hence the $\{1+\ldots\}$ factor in Eq. 9 is set to unity). The "noisy" Weibull model of normalized crests is found to agree well with the observed behavior in all three seastates. As may be expected, the ordinary Weibull model underestimates extreme crests, due to its omission of crest height variability.

Summary and Conclusions

A relatively simple, analytical model of crest distributions has been developed. It utilizes standard, relatively accurate models of the total peak-to-trough wave height. In particular, a Rayleigh model of wave heights has been used; i.e., a Weibull model in Eq. 1 with $\beta_H = 2$ and $\alpha_H = 2\sqrt{2} = 2.82$. One may expect these theoretical values to remain fairly stable, and not need to be reestimated for each new data set.

The remaining three parameters, k_1, k_2, and σ_ε, have been calibrated here to our particular data sets. Note, however, that deterministic nonlinear wave theory (e.g., Stokes) has been found to accurately predict the mean trend $\overline{C}(H)$ (e.g., Jha and Winterstein, 2000). This suggests that k_1 and k_2 can also be estimated directly from theory. This leaves only one parameter, the noise variability, σ_ε, which appears unavailable from theory. Indeed, a critical comparison of data with random Stokes theory (Jha and Winterstein, 2000) suggests that while the theory accurately predicts mean crests and profiles, it systematically underestimates crest variability and hence extreme crest fractiles. This suggests that data be utilized to estimate σ_ε in different cases, to establish the generality of the numerical results cited here.

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References

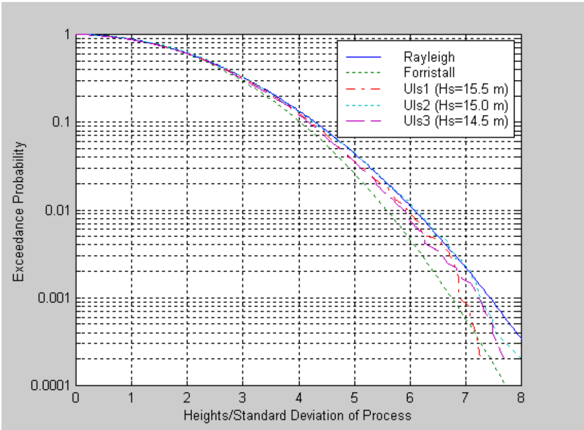
Forristall, G. Z. (1978), "On the Statistical Distribution of Wave Heights in a Storm," *Journal of Geophysical Research*, **83**(C5), 2353-2358.

Haring, R.E., A.R. Osborne, and L.P. Spencer (1976), "Extreme Wave Parameters based on Continental Shelf Storm Wave Records," *Proceedings of the 15th Coastal Engineering Conference*, ASCE, 151-170.

Huang, N.E., S.R. Long, C-C. Tung, Y. Yuan, and L.F. Bliven (1983), "A Non-Gaussian Statistical Model for Surface Elevation on Nonlinear Random Wave Fields," *Journal of Geophysical Research*, **88**(C12), 7597-7606.

Jha, A.K., and S.R. Winterstein (2000), "Nonlinear Random Ocean Waves: Prediction and Comparison with Data," *ETCE/OMAE2000 Joint Conference*, ASME, paper ETCE/OMAE2000-6125.

Tayfun, M.A. (1980), "Narrow-Band Nonlinear Sea Waves," *Journal of Geophysical Research*, **85**(C3), 1548-1552.



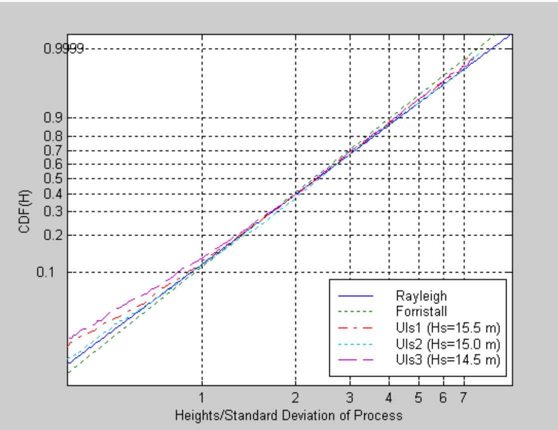
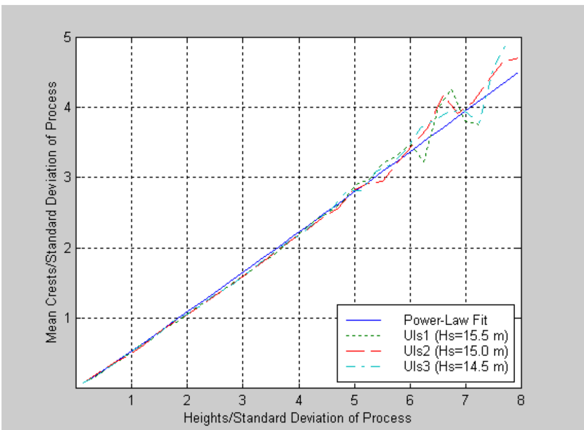


Figure 1 - Exceedance Probability of Heights

Figure 2 – Cumulative Distribution of Heights (Weibull Scale)



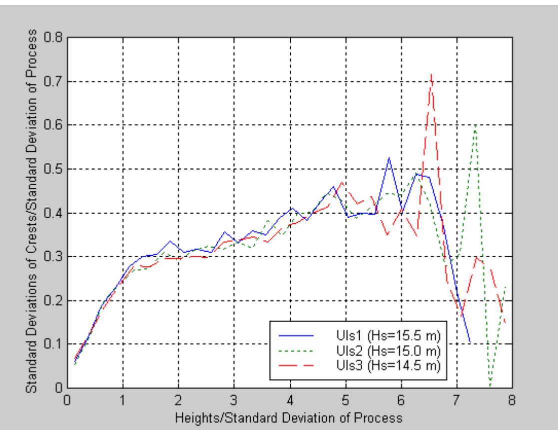
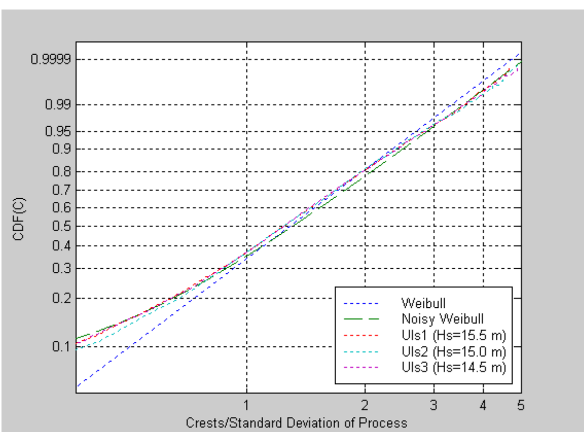


Figure 3 – Cumulative Distribution of Mean Crests Given Heights

Figure 4 – Standard Deviation of Crests Given Heights



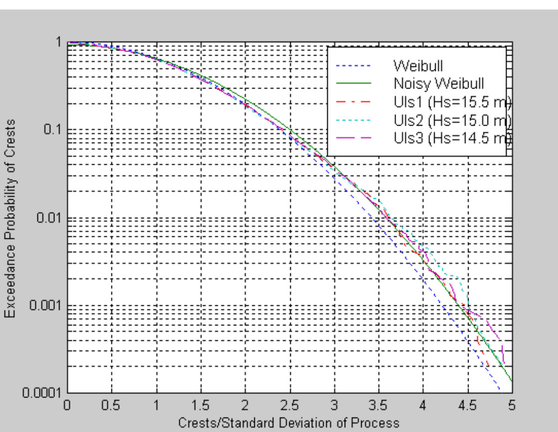


Figure 5 – Cumulative Distribution of Crests (Weibull Scale)

Figure 6 – Exceedance Probability of Crests