Journal of Structural Health Monitoring,
January 1, 2010, Vol. 9, No.~1, 13-24

Efficient Calculation of Statistical Moments for Structural Health Monitoring

Myoungkeun Choi * and Bert Sweetman †

Department of Civil Engineering, Texas A&M University

200 Seawolf Parkway, Galveston, TX 77553-1675, USA

Abstract

Wireless networks of smart sensors with computations distributed over multiple sensor packages have shown considerable promise in providing low-cost Structural Health Monitoring (SHM). In these networks, microprocessors are typically embedded in individual smart sensor packages. The efficiency of embedded computational algorithms is of critical importance because the size, cost, and power requirements of the sensor arrays are central concerns. Here, very efficient methodologies are presented to compute statistical moments of a measured response time-history. These moments: the mean, standard deviation, skewness and kurtosis, are often used to characterize a measured irregular response.

Two alternative approaches are presented, each of which can save substantial computer memory requirements and CPU time in certain applications. The first approach reconsiders the computational benefits of computing statistical moments by separating the data into bins and then computing the moments from the geometry of the resulting

*Myoungkeun Choi: E-mail: mkchoi@tamu.edu

†Bert Sweetman: E-mail: sweetman@tamu.edu, Telephone:(409) 740-4834, Fax:(409) 741-7153

histogram, which effectively becomes a one-pass algorithm for higher moments. One benefit is that the statistical moment calculations can be carried out to arbitrary accuracy such that the computations can be tuned to the precision of the sensor hardware. The second approach is a new analytical methodology to combine statistical moments from individual segments of a time-history such that the resulting overall moments are those of the complete time-history. This methodology could be used to allow for parallel computation of statistical moments with subsequent combination of those moments, or for combination of statistical moments computed at sequential times.

A worked example is presented comparing two implementations of the new methodologies with conventional calculations in monitoring the global performance of an offshore Tension Leg Platform (TLP). Accuracy, efficiency and storage requirements of the calculation methods are compared with those of conventional methods. The results show that substantial CPU and memory savings can be attained with no loss in accuracy and that more dramatic savings can be attained if a slight reduction in accuracy is acceptable.

Keywords Statistical moments, Time series analysis, Structural response, Non-Gaussian, Continuous health monitoring

1 INTRODUCTION

Health monitoring of either old or new structures is desirable for both economic and public safety concerns. The overall goal of these systems is to monitor structural performance such that structural integrity can be regularly assessed to detect any deficiencies before minor structural defects worsen and ultimately result in catastrophic failures. Conventional visual inspection techniques are labor intensive, time-consuming, and costly. Networks of smart sensors that include embedded computers and wireless connectivity have been proposed as a low-cost alternative inspection methodology. Extensive reviews of Structural Health Monitoring (SHM) techniques have been done by others, e.g. [4, 5, 17, 18].

A significant cost to field application of SHM systems on large civil structures can be that of installing the cables; the cabling and its various termination points may also present likely failure points for civil structures in harsh environments such as offshore structures. Finally, some types of structures with rotating components, such as the blades on wind turbines, cabling is nearly impossible. For these reasons and others, active development is ongoing for wireless sensors, often including some degree of distributed computing.

A major challenge for wireless structural monitoring systems is availability of power to run the sensor network for extended periods of time. In most applications, wireless communication between sensors consumes more power than any other operation, which leads designers of these networks to process the raw data locally near the sensor and wirelessly transmit only the results. For example, Lynch [11] embedded damage identification algorithms into wireless sensing units to execute data interrogation. In a sensor network with distributed computing, the wireless sensing units include an embedded microcomputer and are responsible for acquiring sensor measurements, analyzing the measured data, and transmitting the results to a central server or to another sensor package in the network. The computational demands on the embedded microcomputer for real-time local data interrogation can be quite substantial. Employment of very efficient computational algorithms, such as those presented here, will enable lower cost sensors due to reduced CPU time, data storage and power requirements.

Similarly, in a field application of a significant SHM network, long-term storage of actual time-histories may be cost-prohibitive because an enormous amount of data are measured; statistical moments offer a compact way to characterize time-history data for wireless transmission and future storage. The methodologies to combine statistical moments from different segments of a time-history offer a mechanism to combine and reanalyze data to, e.g., detect long-term trends in the underlying process. Sohn et al [14] investigate novelty detection for non-Gaussian response to more accurately assess tail behavior using known extreme-value distributions. Efficient calculation of skewness and kurtosis could enable use of non-Gaussian distributions that more precisely match the tail behavior of measured data, e.g. [24].

In general, statistical moments can be used to represent the characteristics of any random data, e.g., [9, 12]. Statistical moments have found a broad range of application including: blind decomposition [1], asymptotic probability of detection criterion in the frequency domain [6, 7], non-Gaussian noise modeling [22], and condition monitoring and diagnosis of rolling element bearings. Dyer and Stewart first proposed the use of the kurtosis for rolling element bearing defect detection in 1978 [8], and use of statistical moments remains important in this area [10, 13]. These techniques have advantages over traditional time and frequency analysis: lower sensitivity to the variations of load and speed, and easy and convenient analysis of the results [13].

Conventionally, the statistical moments of a set of discrete data, x_i, are computed directly using a two-pass algorithm (e.g. [9, 12]):

$$\mu = \frac{1}{I}\sum_{i=1}^{I} x_i = E[x] = m_1 \tag{1}$$

$$\sigma^2 = \frac{1}{I}\sum_{i=1}^{I}(x_i-\mu)^2 = \theta_2 \tag{2}$$

$$\alpha_3 = \frac{1}{I\sigma^3}\sum_{i=1}^{I}(x_i-\mu)^3 = \frac{\theta_3}{\sigma^3} \tag{3}$$

$$\alpha_4 = \frac{1}{I\sigma^4}\sum_{i=1}^{I}(x_i-\mu)^4 = \frac{\theta_4}{\sigma^4} \tag{4}$$

where I is the number of points in the sample; μ, σ^2, α_3, and α_4 are the mean, variance, skewness, and kurtosis of the data x_i, and θ_2, θ_3 and θ_4 are the central moments. Such algorithms are called two-pass because the mean must first be computed and that mean is subsequently used in the computation of the remaining moments, which implies the entire dataset must be retained. The computational demands imposed by calculation of these moments is a strong function of the number of times a quantity is raised to a power. For example, in calculating the α_4, the quantity $(x_i - \mu)$ is raised to a power of 4, I times:

resulting in one power computation per data-point per statistical moment. Here, computational efficiency of an alternative method to compute statistical moments is re-investigated: the data are first binned to create a histogram from which the desired moments can be calculated. In this method, the number of times a quantity must be raised to a power is a function of the number of bins rather than the number of data points, and the width of each bin can be specified as a function of the required accuracy. Setting the bin width to the precision of the original sensed data yields exact results. The computational savings of this alternative method can be substantial, but perhaps more importantly, dramatic memory savings can also be realized if the raw data are binned real-time, and so the complete time-history does not need to be retained. Avoiding the need to retain a the time-history makes this new methodology competitive with existing one-pass algorithms for the variance, except that here the statistical moments are not computed every time-step. One-pass or on-line algorithms for the mean and variance have been known for some time (e.g., [2]), and have been implemented on real-world hardware (e.g. [21]).

Methodologies for one-pass algorithms for the higher moments, however, are less common. Terriberry [20] offers pairwise updating formulas for the skewness and kurtosis (without derivation). Pebay [15] explains how Terriberry's updating formulae could be implemented as a one-pass algorithm. The one-pass histogram-based algorithm presented here is unique from earlier work in that it allows the user to specify arbitrary accuracy, such that the accuracy of the calculations can be made equivalent to that of the measurement equipment, enabling some savings in computational and memory requirements. Computation of statistical moments from a histogram is generally well-established in the statistics community, but investigation of the methodology as an efficient one-pass algorithm with arbitrarily specified accuracy is unique. As such, this part of the paper is of little theoretical interest to the statistics community, but may be of considerable practical interest to the structural health monitoring community. The technique presented here is less computationally intensive for very large data sets than a true one-pass algorithm because here the data is binned real-time and the moments are computed from the binned data, rather than computing an updated skewness and kurtosis every time-step. The Terriberry/Pebay one-pass algorithm is derived as a special case of a pairwise updating algorithm, which is substantially different than the derivation presented here.

In a related calculation, it is often convenient to use an updating algorithm to combine statistical moments from individual segments of a data set. This combination can be useful such that various segments of the data can be processed in parallel, or such that segments of a time-history can be processed sequentially as they are measured. In SHM problems, it can be useful to compute statistical moments of new measured data for comparison with those of previously measured historic data. If no differences are observed, then the new data are merged into the historic data to give a more robust estimate of the true underlying moments. The most simple, though inefficient, way to revise the historic statistical moments would be to concatenate the new data onto the old, then compute the moments of the complete history. Here, a new pairwise method is presented to combine the statistical moments computed from individual segments of a time-history, without the need to recompute any of these statistical moments. Using this technique, only five historic values need to be retained: the first four statistical moments and the amount of data on which they are based.

In addition to explaining the use of Terriberry's [20] results as a one-pass algorithm, Pebay [15] also notes that Terriberry's results are special cases of Pebay's arbitrary-order update formulae. The implementation offered by Pebay is substantially different from that suggested here, and the derivation offered by Pebay also differs meaningfully from that offered here, though both derivations hinge on the commutativity of summations over finite sets as applied to statistical moments. In his report, Pebay echoes Terriberry's thoughts, noting that: "To our knowledge, there are currently no published formulas for parallel updates of higher-order moments." An important capability of the updating formulae offered here compared with those of Terriberry/Pebay is that this methodology can be readily applied to distributions of data that are not specifically countable, such as the moments of a probability distribution estimated from e.g. a power spectrum (e.g., [19]). Also of potential interest to the structural health monitoring community is that this new methodology is easily modified to allow a user to assign different importance to specific segments of the data, such as newer data being more important than older data, or one sensor being more reliable or accurate than another. Finally, the Terriberry formulae apply to combination of only two sets of higher-order moments, and would need to be used recursively to combine multiple sets of moments. In contrast, the new methodology presented here can be used to combine any number of sets of moments, which could be relevant in the case of massively parallel computations.

2 THEORY

Derivation of the new methodology requires some knowledge of moments, which is presented here to ensure consistent notation. The background is followed by an investigation of computing statistical moments of discrete data from a histogram, with emphasis on the relative efficiency and accuracy compared with conventional methods. Finally, the proposed method to combine statistical moments is derived and presented. Throughout most of this text, the data set is referred to as an irregular time-history because that is the most common application to structural health monitoring. In general, the equations and techniques presented here are equally valid for any random variable.

2.1 Background

2.1.1 Calculation of moments from discrete data

The moments of a random variable about zero and about its mean are referred to as the raw and central moments, respectively. The n^{th} moment of a discrete random variable $x(t)$

about value r with a finite range is defined as (e.g. [9, 12]):

$$M_{n,r} = \sum_{i=1}^{I} \Big(x(t_i) - r\Big)^n \Delta t_i \tag{5}$$

which can be expressed as follows if the continuous distribution function, $f(x)$, is known:

$$M_{n,r} = \sum_{k=1}^{K} (x_k - r)^n f(x_k) \Delta x_k \tag{6}$$

in which I represents the number of data points $x(t_i)$ and K represents the number of base x_k in its discrete distribution function $f(x_k)$, i.e., the number of bins in the discrete distribution.

The n^{th} raw moment of a discrete time-history (Equation (5) with $r = 0$) can be normalized by the time duration, with the result equal to the expected value of the n^{th} power of x

$$m_n^{(t)} = \frac{1}{T} M_{n,0} = \frac{\sum_{i=1}^{I} x(t_i)^n \Delta t_i}{\sum_{i=1}^{I} \Delta t_i} = E[x^n] \tag{7}$$

where $T = \sum_{i=1}^{I} \Delta t_i$ is the time duration and the superscript $^{(t)}$ indicates moments are calculated directly from the time-history. For constant Δt, the duration is $T = I\Delta t$, which enables Equation (7) to be simplified:

$$m_n^{(t)} = \frac{1}{I} \sum_{i=1}^{I} x(t_i)^n \tag{8}$$

The first normalized raw moment ($n = 1$) is the sample mean, which is often used to estimate the true mean of the process for normalization of other central moments.

$$\theta_n^{(t)} = \frac{1}{T} M_{n,m_1^{(t)}} = \frac{\sum_{i=1}^{I} \Big(x(t_i) - m_1^{(t)}\Big)^n \Delta t_i}{\sum_{i=1}^{I} \Delta t_i} = E\Big[\Big(x - m_1^{(t)}\Big)^n\Big] \tag{9}$$

which for constant time interval Δt is:

$$\theta_n^{(t)} = \frac{1}{I} \sum_{i=1}^{I} \Big(x(t_i) - m_1^{(t)}\Big)^n \tag{10}$$

The second normalized central moment ($n = 2$) is the sample variance (Equation (2)).

2.1.2 The relationship between raw moments and central moments

The first four raw moments and central moments have the following well-known mathematical relationships, e.g. [9, 12]:

$$m_1 = E[x] = \mu \tag{11}$$

$$m_2 = E[x^2] = \theta_2 + m_1^2 \tag{12}$$

$$m_3 = E[x^3] = \theta_3 + 3m_1\theta_2 + m_1^3 \tag{13}$$

$$m_4 = E[x^4] = \theta_4 + 4m_1\theta_3 + 6m_1^2\theta_2 + m_1^4 \tag{14}$$

$$\theta_2 = E[(x-\mu)^2] = m_2 - m_1^2 \tag{15}$$

$$\theta_3 = E[(x-\mu)^3] = m_3 - 3m_1m_2 + 2m_1^3 \tag{16}$$

$$\theta_4 = E[(x-\mu)^4] = m_4 - 4m_1m_3 + 6m_1^2m_2 - 3m_1^4 \tag{17}$$

2.2 Calculation of Moments using a Relative Histogram

A relative histogram of a random variable can be constructed in the conventional way. The range of potential values is divided into bins and the number of occurrences within each bin are counted and plotted such that the area of each rectangle equals the portion of the sample values within that bin (e.g. [9, 12]):

$$H(x_k) = \frac{h(x_k)}{A} \tag{18}$$

where $h(x_k)$ and $H(x_k)$ represent the frequency and the relative frequency at bin x_k, and $A = \sum_{k=1}^{K} h(x_k)\,\Delta x_k$ is the total area of the histogram. After this normalization, the n raw moments and central moments of $x(t)$ can be calculated from the relative histogram, similar to Equation (6):

$$m_n^{(h)} = \sum_{k=1}^{K} x_k^n\, H(x_k)\Delta x_k = \frac{1}{A}\sum_{k=1}^{K} x_k^n\, h(x_k)\Delta x_k \tag{19}$$

$$\theta_n^{(h)} = \sum_{k=1}^{K} \left(x_k - m_1^{(h)}\right)^n H(x_k)\Delta x_k = \frac{1}{A}\sum_{k=1}^{K} \left(x_k - m_1^{(h)}\right)^n h(x_k)\Delta x_k \tag{20}$$

where the superscript $^{(h)}$ indicates the moments are calculated from the histogram. For constant bin width $\Delta x_k = \Delta x$ these two expressions can be simplified with $A = \sum_{k=1}^{K} h(x_k)\Delta x = I\Delta x$:

$$m_n^{(h)} = \frac{1}{I}\sum_{k=1}^{K} x_k^n\, h(x_k) \tag{21}$$

$$\theta_n^{(h)} = \frac{1}{I}\sum_{k=1}^{K} \left(x_k - m_1^{(h)}\right)^n h(x_k) \tag{22}$$

2.2.1 Precision of moments calculated from a histogram

Calculation of sample moments directly from the time-history results in optimal accuracy since all data are used directly and there is virtually no opportunity for round-off error. The histogram method also yields perfect accuracy if the bin width is equal to the precision with which the data has been measured. If computational efficiency is of greater importance than absolute precision, then the bins can be made arbitrarily wider, trading a reduction in computational demands for reduced accuracy. The resulting difference between moments calculated using a time-history and using its histogram represents the error:

$$m_n^{(h)} - m_n^{(t)} = \frac{1}{A}\sum_{k=1}^{K} x_k^n\, h(x_k)\Delta x_k - \frac{1}{T}\sum_{i=1}^{I} x(t_i)^n \Delta t_i \tag{23}$$

which for both constant time interval Δt and constant bin width Δx are:

$$m_n^{(h)} - m_n^{(t)} = \frac{1}{I}\left(\sum_{k=1}^{K} x_k^n\, h(x_k) - \sum_{i=1}^{I} x(t_i)^n\right) \tag{24}$$

The trade-off between precision and computational savings is investigated as part of the example later in this paper.

2.3 Efficient Combination of Statistical Moments

Here, a new computational method is proposed to combine multiple sets of statistical moments. An example application would be combining moments from several individual segments of a long time-history, with each segment possibly being processed by a separate processor. If sample statistical moments describing several separate segments of an irregular time-history have been computed from measured data, statistical moments describing a single concatenated time-history of all data can be calculated directly from the existing statistical moments. This proposed computational technique uses the first four statistical moments of each segment to compute the four raw moments, which are then transformed into new variables (γ_n) that are easily combined by addition. After combination, the new variables are inversely transformed back to four raw moments now describing all the data, from which the statistical moments are easily calculated.

2.3.1 Moment addends, γ_n

New moment addend variables, γ_n, are introduced to enable straightforward combination of the statistical moments of multiple time-histories. For an irregular time-history $x(t)$ with variable time interval Δt_i:

$$\gamma_n = \sum_{i=1}^{I} (x(t_i))^n \Delta t_i \tag{25}$$

where γ_0 is the duration of each time-history. For constant time interval $\Delta t_i = \Delta t$:

$$\gamma_n = \Delta t \sum_{i=1}^{I} (x(t_i))^n \tag{26}$$

The same values of γ for the histogram form of computing the moments can be expressed in terms of the frequency of occurrence at the x_k bin, $h(x_k)$, with variable bin width Δx_k.

$$\gamma_n = \sum_{k=1}^{K} x_k^n h(x_k) \Delta x_k \tag{27}$$

yielding γ_0 as the area of the histogram. For constant bin width $\Delta x_k = \Delta x$:

$$\gamma_n = \Delta x \sum_{k=1}^{K} x_k^n h(x_k) \tag{28}$$

2.3.2 Combination of statistical moments

If Q sets of statistical moments are known: $(\gamma_{0,q}, \mu_q, \sigma_q^2, \alpha_{3,q}, \alpha_{4,q}) \quad \text{for} \quad q = 1, 2, ..., Q$, then each γ_n can be expressed in terms of the equivalent n raw moments (Equations 7, 8, 19, and 21).

$$\gamma_{n,q} = m_{n,q}\gamma_{0,q} \qquad \text{for} \quad n = 1, 2, 3, 4 \quad \text{and} \quad q = 1, 2, ..., Q \tag{29}$$

where $\gamma_{0,q}$ is generally taken to be the duration of the q^{th} time-history, or the number of points if Δt is constant. It is worth noting, however, that $\gamma_{0,q}$ is a weighting factor only, and its interpretation can be flexible depending on the application. Importantly, in this method the statistical moments are not required to be those of a quantity that is countable: these moments could be computed directly from e.g., a probability distribution, in which case the value of $\gamma_{0,q}$ would represent the relative importance of the moment estimate. There is no theoretical limitation on the maximum order of the moments (the value of n), though higher-order equivalents to Equations (11–17) would be needed in practical applications. The benefit of expressing the statistical moments in terms of γ is that the Q sets can be combined by addition, and there is no upper limit on the value of Q.

$$\gamma_{n,c} = \sum_{q=1}^{Q} \gamma_{n,q} \qquad \text{for} \quad n = 0, 1, 2, 3, 4 \tag{30}$$

where the subscript $_c$ represents the concatenated time-history or combined γ. These combined values of γ can then be inversely transformed into raw moments representing the concatenated time-history by inverting Equation (29).

$$m_{n,c} = \frac{\gamma_{n,c}}{\gamma_{0,c}} \quad \text{for} \quad n = 1, 2, 3, 4 \tag{31}$$

The relationship between raw moments and central moments (Equations 15–17) are then used to compute the central moments of the concatenated time-history. Finally, the statistical moments of the concatenated history are computed as in Equations (1–4)

$$\mu_c = m_{1,c} \qquad \sigma_c^2 = \theta_{2,c} \qquad \alpha_{3,c} = \frac{\theta_{3,c}}{\sigma_c^3} \qquad \alpha_{4,c} = \frac{\theta_{4,c}}{\sigma_c^4} \tag{32}$$

3 APPLICATION

3.1 Estimation of Statistical Moments of a Concatenated Time-history

In general, sample moments approach the true moments of the underlying process as the time-history increases in length. If very long time-histories are available, moments of the true process can often be estimated by those of the long sample. However, if individual segments of the long history are available, several possible approaches to finding moments of the concatenated history are available. Three of these possibilities are presented in Figure 1. Each of these methods can be applied repeatedly to compute statistics of a long time-history made up of any number of segments.

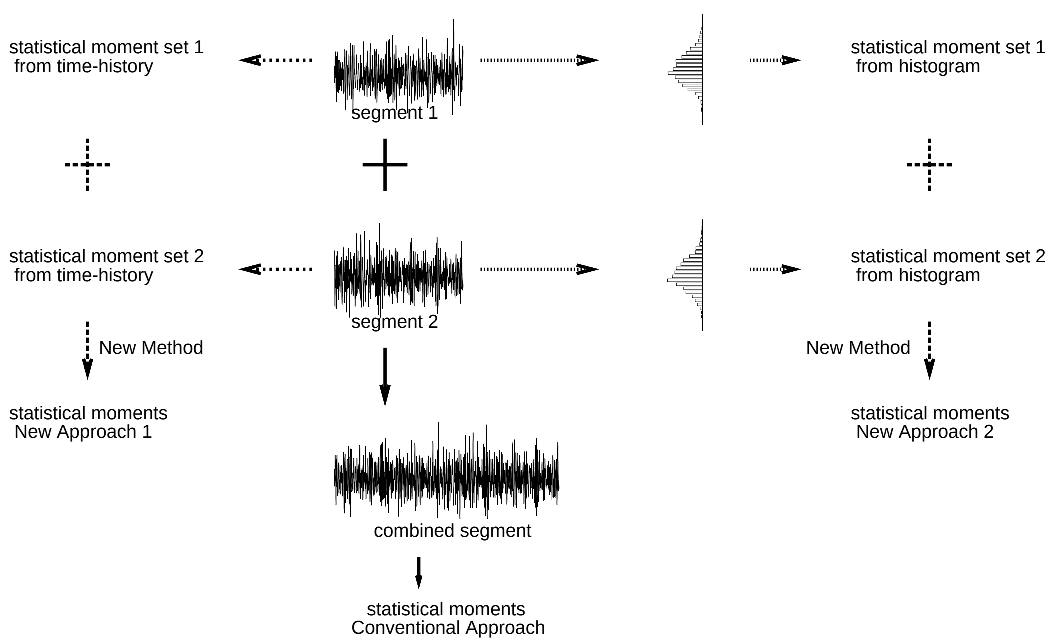

Figure 1: Three Approaches to Calculate Statistical Moments of a Concatenated Time-History

The most conventional method is shown in the center of the figure: the first segment and the second segment are concatenated and the statistical moments are calculated directly from the resulting long time-history. This simple approach requires considerable computational resources and substantial storage space to retain the previous time-histories, both of which may be problematic in field applications using micro-computers in distributed sensor arrays.

Two alternative approaches are shown progressing down the left and right sides of Figure 1. The principle benefit compared with the conventional approach is a significant decrease in storage requirements since only the statistical moments need to be retained, rather than the complete time-histories. In both cases, the methods of Section 2.3.2 are applied to combine these moments. The two approaches differ only in how the statistical moments for each segment are computed before concatenation. The method on the left side of the figure is based on calculation of the statistical moments of each time-history by the conventional method of Equations 1–4; the approach shown on the right side of the figure is based on calculation of the statistical moments through use of the histogram as in Section 2.2.

3.1.1 Calculation of statistical moments from a histogram

One method to calculate the statistical moments is by calculation of the central moments from the histogram (Equations 20 or 22) and then converting these to the central statistical moments through the relationships of (Equations 1–4). If the bin width of the histogram is equivalent to the smallest decimal place of the measured data, then the results will be identical to those calculated by conventional means. As previously noted, for very long time-histories the histogram method will generally be more efficient; for shorter time-histories the more conventional method will generally be more efficient. However, the histogram method also offers an additional option to reduce computational demands: the bin width can be made larger than the smallest decimal place of the measured data. The effects of increased bin width versus computational demands are investigated in a later example.

3.1.2 Combination of statistical moments

Regardless of how the statistical moments of individual segments of a time-history were calculated, these moments can be combined efficiently. The procedure is straightforward: First, the $n = 4$ statistical moments for each (q) of the Q segments to be combined are transformed into the mean and three central moments by inverting the definitions of the statistical moments (Equations 1–4) to $m_{1,q} = \mu_q$, $\theta_{2,q} = \sigma_q^2$, $\theta_{3,q} = \alpha_{3,q}\sigma_q^3$, $\theta_{4,q} = \alpha_{4,q}\sigma_q^4$. Second, the resulting Q sets of three central moments are transformed into Q sets of three raw moments using the well-known relationships between raw and central moments (Equations 12–14). Third, the resulting $4Q$ raw moments (including Q means) are then transformed into $4Q$ values of $\gamma_{n,q} = m_{n,q}\gamma_{0,q}$ (Equation (29)). Fourth, each of the 5 sets of Q values of $\gamma_{n,q}$ are combined as in Equation (30), $\gamma_{n,c} = \sum_{q=1}^{Q} \gamma_{n,q}$ ($n = 0$ to 4). Finally, the transformation process is reversed for the resulting 5 values of $\gamma_{n,c}$ to produce the desired four central statistical moments as in Equations 31–32.

4 EXAMPLE

The new methodologies are applied to simulated data resulting from a time-domain solution of a simple numerical model of a Tension Leg Platform subject to irregular seas. A TLP was selected for this example because of its highly non-Gaussian surge response (horizontal translation in the direction of the environmental loading). Three segments of a surge time-history, each having significantly different statistical moments, are created without modifying the structural model, but only changing the peak period of the incident waves, the current velocity and the wind force. Calculation and combination of the resulting statistical moments are investigated and compared.

4.1 Numerical Simulations

A Tension Leg Platform (TLP) is a compliant offshore structure used for production of oil and gas in deep ocean waters. The platform is vertically moored by tendons at each of its corners (Figure 2). Surge response time-histories of the Snorre TLP, which is located in the Norweigian sector of the North Sea, have been simulated by a simplified 2-dimensional nonlinear numerical model.

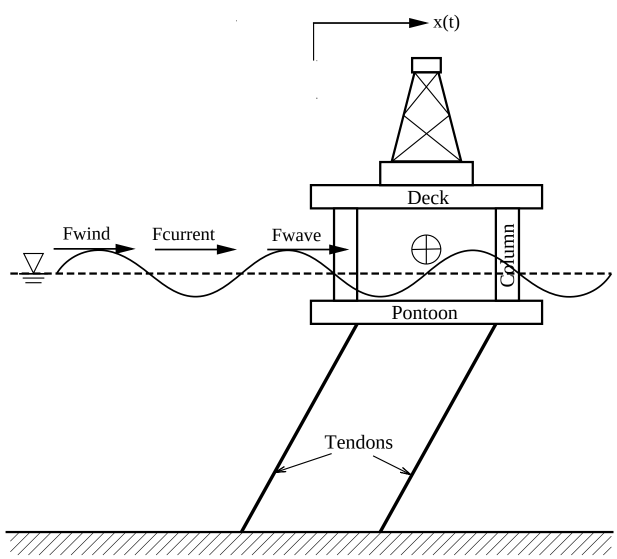

Figure 2: Tension Leg Platform

The wave and current forces are non-linear and non-Gaussian. These forces are applied to a single degree of freedom system including nonlinear restoring force. The equation of motion is solved in the time domain using the Newmark Beta Method (e.g., [3]). The time step of integration (Δt) is 0.01 sec and total time duration for each of the three time-histories is one hour. The two main sources of non-Gaussianity in the model are 1) the non-linear mooring restoring force caused by the changing angles of the tendons with increased offset and the increased buoyancy of the hull caused by being pulled downward by the tendons, and 2) the non-Gaussian wave forcing caused by the highly nonlinear drag term in the Morison Equation [16, 23].

4.2 Results

The statistical moments of each response time-history simulated under three environmental conditions are summarized in Table 1. Changing environmental conditions result in substantial changes in the statistical moments of the simulated response.

4.2.1 Calculation of statistical moments

Calculation of statistical moments by the conventional methods of Equations 1–4 are compared with application of the relative histogram (Section 3.1.1) in Figure 3. The vertical axis on the left compares the relative CPU time needed to compute all four of the statistical moments, with the conventional methodology defined as 100% CPU time. For this one-hour time-history (360,000 data points) with a measurement precision of 0.001, the histogram method takes about 75% of the CPU time as the conventional method if the bin width is set to the precision (zero error in binning leading to exact sample statistical moments). CPU usage figures result from binning and computations performed using MatLab.

Progressing from left to right on the horizontal axis shows gradual increases in the bin width, and the vertical axis on the right shows the associated error in the statistical moments. The error is the difference between the exact and approximate statistical moments divided by the exact statistical moments (calculated conventionally). The plot shows that for this time-history, increasing the bin width to ten times as large as the precision of the data has a savings of about half of the CPU time, with virtually no noticeable increase in error in the statistical moments. The CPU time does not drop by a factor of ten because the same 360,000 data points must be binned.

Calculation using a histogram with bin width of 0.001 yields results identical to the conventional method (Table 1), with using 74.7% of the CPU time and 2.78% of the storage requirements. Calculation using a histogram with bin width of 0.01 results in statistical moments with error less than 0.1% using 33.9% CPU time and 0.25% storage space relative to conventional calculation. The dramatic savings in storage space results from assuming the data are binned as they are collected such that the time-history need not be saved. Computation of statistical moments by the conventional two-pass method requires saving the complete time-history so the mean can be computed in the first pass and applied to computation of the moments in the second pass through the data.

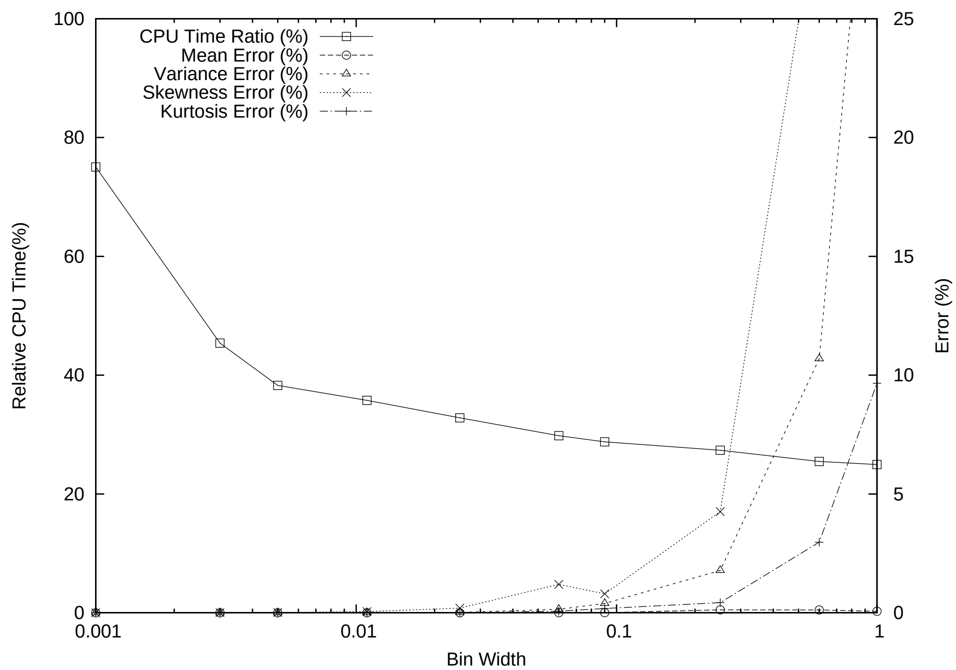

Figure 3: Effect of bin width on calculation of statistical moments. Time duration: 1 hour, Δt: 0.01 sec, Data precision: 0.001

4.2.2 Calculation of statistical moments of the concatenated time-history

Three approaches are compared for calculating the statistical moments of a concatenated time-history made up of the three independent time-histories described by Table 1.

Method A (Conventional): The conventional method of calculating the statistical moments defines 100% CPU and 100% Storage requirements. This method is to store both of two one-hour time-histories as they are collected, to compute the statistical moments using the conventional means of Equations 1–4, then to concatenate the two histories, and then finally to compute the statistical moments of the concatenated time-history.

Method B (Statistical Concatenation): In this application, it is assumed that the first time-history is stored and conventional methods are applied to calculate the statistical moments; the first time-history is then deleted and the second is stored and its statistical moments are calculated. These two sets of statistical moments are then combined using the methods of Section 3.1.2. The approximately 50% savings in both CPU usage and memory requirements results from not having to save both time-histories and recompute the moments for the concatenated history.

Method C (Histogram with Statistical Concatenation): The method applied here is to calculate the moments using both the histogram method of Section 3.1.1 and the statistical method of Section 3.1.2 to combine moments. Results from Method (C) are presented both for $\Delta x = 0.001$, which yields exact results, and for $\Delta x = 0.01$, which introduces some error in return for greater CPU and storage savings. Method (C) with adequate bin width for an exact solution requires only about 27% of the CPU time and around 1.4% of the required memory. The dramatic memory savings result from the assumption that the time-history will be binned on the smart sensor package as the data are collected, such that no time-history ever needs to be stored. Greater savings are also shown if the bin width is made ten times as large, though the resulting moments are not exact.

Methods (D), (E) and (F) are equivalent, but the single concatenated segment resulting from (A), (B) and (C) is now combined with the third segment.

5 CONCLUSIONS

Two methodologies have been presented and demonstrated through an example. The first methodology is effectively a single-pass methodology for computation of higher statistical moments through use of a histogram. This methodology is based on a simple combination of methods well-known in the statistical community and is therefore of little theoretical interest. It may, however, be of considerable practical interest in the field of structural health monitoring because it enables computation of the skewness and kurtosis to arbitrarily selected accuracy, offering a means to effectively trade computational intensity against accuracy. Generally, for very long time-histories with low-precision data it is more efficient to use the histogram for statistical moment calculation since the number of power calculations is smaller, but for relatively short time-histories with high-precision data the more conventional method will require less CPU time.

The second methodology enables direct combination of the skewness and kurtosis of any number of data sets. The method and its derivation are new, and may be of theoretical interest. One unique aspect of this new methodology is that it can be used to combine statistical moments of data that is not countable, e.g., moments extracted directly from a probability density spectrum.

An example is presented in which both the histogram approach and the updating methodology for the skewness and kurtosis are verified. The relative efficiency and accuracy of the histogram approach are examined in some detail. In the example, statistical moments for time-histories are computed using both the conventional approach and the histogram approach, including setting varying degrees of accuracy. Setting the bin width equal to the precision of the measured data gives perfect accuracy and potential computational savings. A noticeable decrease in computational demands accompanied by some decrease in accuracy caused by increasing the bin width is shown in Figure 3. In the example, CPU savings on the order of 75% of that required for a conventional two-pass algorithm are realized with only a minor decrease in accuracy. Savings in memory requirements can also be quite substantial (Table 2). In the example, applying the histogram as a one-pass algorithm by binning the data as it is collected and then computing the moments from the resulting histogram uses around 1% of the memory compared with a more conventional two-pass algorithm.

6 Acknowledgements

This work was supported by the National Science Foundation, Division of Civil and Mechanical Systems under Agreement Numbers CMS-0428585 and CMS-0448730. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the view of the National Science Foundation.

Condition	$H_s(m)$	$T_p(sec)$	$U_c(m/s)$	$F_w(kN)$	μ	σ^2	α_3	α_4
1	14.5	15.5	1.5	-3000	-1.3833	0.2742	0.1236	2.9969
2	14.5	12.5	2.0	-5000	-2.2512	0.2447	0.1203	2.3685
3	14.5	18.5	1.0	-2000	-0.9154	0.2510	0.0411	2.1207

Table 1: Environmental conditions and associated statistics of the response

Method	Segment	μ	σ^2	α_3	α_4	CPU(%)	Storage(%)
Conventional (A)	1+2	-1.8173	0.4478	0.1180	2.5684	100.00	100.00
Statistical	1+2	-1.8173	0.4478	0.1180	2.5684	51.86	50.00
Concatenation (B)	Error(%)	0.0000	0.0000	0.0000	0.0000	-	-
Histogram w/	$\Delta x = 0.001$	-1.8173	0.4478	0.1180	2.5684	26.58	1.39
Stat. Concat. (C)	Error(%)	0.0000	0.0000	0.0000	0.0000	-	-
	$\Delta x = 0.01$	-1.8177	0.4478	0.1183	2.5687	20.86	0.14
	Error(%)	0.0021	0.0054	0.0199	0.0034	-	-
Conventional (D)	12+3	-1.5167	0.5629	-0.0917	2.3856	100.00	100.00
Statistical	12+3	-1.5167	0.5629	-0.0917	2.3856	34.99	33.33
Concatenation (E)	Error(%)	0.0000	0.0000	0.0000	0.0000	-	-
Histogram w/	$\Delta x = 0.001$	-1.5167	0.5629	-0.0917	2.3856	15.63	0.93
Stat. Concat. (F)	Error(%)	0.0000	0.0000	0.0000	0.0000	-	-
	$\Delta x = 0.01$	-1.5171	0.5630	-0.0913	2.3856	11.82	0.09
	Error(%)	0.0279	0.0185	-0.3742	-0.0008	-	-

Table 2: Statistical moments of concatenated irregular TLP response time-histories; comparison of calculation methodologies

Nomenclature

x_i **or** $x(t_i)$ Individual point in an irregular data set or time-history

I The number of data points in a data set

Δt_i **or** Δt Time interval between times t_i and t_{i+1} or constant interval between data points

T Total duration of a time history

x_k Value of k^{th} bin in a histogram of random data

Δx_k **or** Δx Bin width of bin x_k or constant bin width

$h(x_k)$ **and** $H(x_k)$ Absolute and relative occurrences of data points in bin x_k

K The total number of bins

$M_{n,a}$ n^{th} moment of irregular data about a

m_n n^{th} normalized raw moment of irregular data

θ_n n^{th} normalized central moment of irregular data

μ Mean of a data set

σ^2 Second statistical moment of irregular data: Variance of the process

α_3 Third statistical moment of irregular data: Skewness of the process

α_4 Fourth statistical moment of irregular data: Kurtosis of the process

$_{,q}$ subscript representing the q^{th} data set or time history

$_{,c}$ subscript representing combined statistics of concatenated data

Q The total number of data sets

γ_n The n^{th} value of γ, used in combination of statistical moments

References

1. Cadzow, J. A. and Li, X. (1995). Blind decomposition. *Digital Signal Processing*, 5:3–20.

2. Chan, T. F., Golube, G. H., and LeVeque, R. J. (1979). Updating formulae and a pairwise algorithm for computing sample variances. Report STAN-CS-79-773, Department of Computer Science, Stanford University.

3. Chopra, A. K. (2001). *Dynamics of Structures: Theory and Applications to Earthquake Engineering*. Prentice Hall.

4. Doebling, S., Farrar, C., and Prime, M. (1998). A summary review of vibration-based damage identification methods. *The Shock and Vibration Digest*, 30(2):91–105.

5. Doebling, S. W., Farrar, C. R., Prime, M. B., and Shevitz, D. W. (1996). Damage identification and health monitoring of structural and mechanical systems from changes in their vibration characteristics: A literature review. Technical Report LA-13070-MS, Los Alamos National Laboratory, Los Alamos, New Mexico 87545.

6. Dwyer, R. (1984). Use of the kurtosis statistics in the frequency domain as an aid in detecting random signals. *IEEE Journal of Oceanic Engineering*, 9:85–92.

7. Dwyer, R. (1985). Asymptotic detection performance of discrete power and higher-order spectra estimates. *IEEE Journal of Oceanic Engineering*, 10:303 – 315.

8. Dyer, D. and Stewart, R. M. (1978). Detection of rolling element bearing damage by statistical analysis. *ASME Journal of Mechanical Design*, 100:229235.

9. Ehrenfeld, S. and Littauer, S. B. (1964). *Introduction to Statistical Methods*. McGraw-Hill.

10. Heng, R. B. W. and Nor, M. J. M. (1998). Statistical analysis of sound and vibration signals for monitoring rolling element bearing condition. *Applied Acoustics*, 53:211–226.

11. Lynch, J. P., Sundararajan, A., Law, K. H., Kiremidjian, A. S., and Carryer, E. (2004). Embedding damage detection algorithms in a wireless sensing unit for operational power efficiency. *Smart Materials and Structures*, 13:800–810.

12. Mack, C. (1967). *Essentials of Statistics for Scientists and Technologists*. Plenum Press.

13. Martin, H. R. and Honarvar, F. (1995). Application of statistical moments to bearing failure detection. *Applied Acoustics*, 44(1):67–77.

14. Parka, H. W. and Sohn, H. (2006). Parameter estimation of the generalized extreme value distribution for structural health monitoring. *Probabilistic Engineering Mechanics*, 21:366–376.

15. Pebay, P. (2008). Formulas for robust, one-pass parallel computation of covariances and arbitrary-order statistical moments. Report SAND2008-6212, Sandia National Laboratories.

16. Sarpkaya, T. and Issacson, M. (1981). *Mechanics of Wave Forces on Offshore Structures*. Van Nostrand Reinhold.

17. Sohn, H., Farrar, C., Hunter, N., and Worden, K. (2001). Structural health monitoring using statistical pattern recognition techniques. *Journal of Dynamic Systems, Measurement, and Control*, 123:706–711.

18. Sohn, H., Farrar, C. R., Hemez, F. M., Shunk, D. D., Stinemates, D. W., and Nadler, B. R. (2003). A review of structural health monitoring literature: 1996-2001. Report LA-13976-MS, Los Alamos National Laboratory.

19. Sweetman, B. and Choi, M. (2006). The modal distribution method: A new statistical algorithm for analyzing measured data. In *Proceedings of SPIE*.

20. T. B. Terriberry (2008). Computing higher-order moments online. [online] http://people.xiph.org/ tterribe/notes/homs.html.

21. Tanner, N. A., Wait, J. R., Farrar, C. R., and Sohn, H. (2003). Structural health monitoring using modular wireless sensors. *Journal of Intelligent Material Systems and Structures*, 14:43–56.

22. Tesei, A. and Regazzoni, C. (1995). Signal detection in non-gaussian noise by a kurtosis-based probability density function model.

23. Wilson, J. F., editor (2002). *Dynamics of Offshore Structures*. John Wiley & Sons.

24. Winterstein, S. R. (1988). Nonlinear vibration models for extremes and fatigue. *Journal of Engineering Mechanics*, 114.