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The Momentum Cloud Method for Dynamic Simulation of Rigid Body Systems

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Abstract

A new formulation of multibody dynamics, the momentum cloud method (MCM), is presented. The method is based on applying the conservation of momentum directly to a complete system, and makes use of prescribed motions between each contiguous body. The absolute translation and rotation of a prescribed base body within the system are chosen to be reference point coordinates. The relative rotations between contiguous bodies along the kinematic chain within the system are chosen to be relative coordinates, the effect of which are captured as a series of cascading transformation matrices. The final result is that the motion of an *N*-body system can be conveniently represented using only six equations of motion (EOMs). Numerical integration of these EOMs is facilitated by representing the mass matrix of the entire system as two 3×3 matrices. The solutions to three coupled rotational EOMs based on conservation of angular momentum of the system are Euler angles describing the rotation of the base body. The solutions to three coupled translational EOMs based on conservation of linear momentum of the system are the translation of the system measured at its center of mass (CM), which are then transferred into translation of base body. Additionally, the inverse dynamic analysis can be used to obtain internal forcing between any two contiguous bodies. The new method is derived for a generalized serial *N*-body system connected by revolute joints with prescribed relative rotation, and then expanded to more complicated forms and joints. A simulation example is presented for a 6-body floating wind turbine system.

CE Database subject headings: Simulation, Structural dynamics, Rigid-body dynamics, Wind power, Floating structures

1. Introduction and Background

A multibody system is defined as an assembly of two or more rigid bodies joined together and having the possibility of relative movement between them (Jalon and Bayo, 1994). Dynamic simulation of multibody systems has broad applicability in engineering, including application in robotics, industrial machinery, aerospace, and automobile systems. Prior work has been done to simulate dynamic systems subject to large-amplitude displacements. For example, Stoneking (Stoneking, 2007) presents the derivation of the exact nonlinear dynamic EOMs for a multibody

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spacecraft connected by spherical gimbal joints. Dynamics of robots are systematically solved by Mason (Mason, 2001); Kurfess (Kurfess, 2004) systematically models the dynamics of robots using conventional methods of formulation of equations of motion. Featherstone (Featherstone, 2008) investigates the dynamics formulation of a floating-base rigid-body system, in which the base body is free to move in the space.

There are various classical analytical methods for the establishment of the EOMs in dynamic analysis of multibody systems, generally based on momentum theory, such as Newton-Euler (NE) method (1750), or energy theory, such as Euler-Lagrange (EL) method (1788) and Kane's method (1985). Many books summarize conventional analytical techniques of multibody dynamics and the generalized formulations of EOMs (Shabana, 1989; Huston, 1990; Kurfess, 2004). Essentially all methods for obtaining the EOMs are equivalent, but the application scope may differ from each other in terms of specific problems (Luca, 2000). The NE equations are usually established by separating the free-body diagrams of each rigid body in the system. A key advantage of the NE method is that the effect of newly added body on the EOMs can be conveniently represented by a recursive formulation procedure. A key disadvantage is that the internal forcing at each joint must be considered to solve the EOMs of the system, which may not be efficient if only a few internal forces are concerned. The EL method avoids the calculation of the internal forcing, which do not perform work. However, that method requires derivation of partial derivatives of energy with respect to related DOFs, which can be laborious. Kane's method combines the advantages of the previous two methods and enables the user to formulate the EOMs through application of virtual power theory, which avoids the calculation of internal forcing and the differentiation of energy functions, but still requires laborious rederivation of the EOMs when a new body with additional DOFs is added to the system. The new method presented here (the MCM) derives closed-form EOMs by applying the conservation of momentum to the multibody system directly. This formulation is a fundamental departure from current analytical methods for the multibody problem: it combines the advantages of energy and momentum approaches in that it enables motion simulations without the computational demands and potential numerical problems associated with calculating internal forcing, and it enables expansion to include additional bodies without reformulation. Additionally, it enables reduction of simulation of an N-body system with a system of only six dynamic equations, regardless of dimension N.

Regardless of the physical theories, the general form of the EOMs is based on the representation of a mass matrix for use in computing the inertial forcing, which is then set equal to the external forcing. The EOMs in this general form are commonly rewritten in the first-order decoupled form $\dot{x} = f(x, t)$ for convenient numerical integration (Meghdari and Fahimi, 2001). Featherstone (Featherstone, 2008) investigates the formulation and solution of the EOMs in this general form from the perspective of coding. Orden (Orden et al., 2010) analyzes the computational methods and applications of multibody dynamics. Some programs based on analytical methods have also been documented in the literature. The computational demands associated with numerical integration of the equations of motion is increased by coupling between elements in the mass matrix, which appear as nonzero off-diagonal elements. In this sense, the formulation of existing commonly used methods introduces a computational inefficiency: a large number of coupled differential EOMs must be solved simultaneously. Various researchers have developed methods to overcome these inefficiencies, but the approach here is to avoid them by decoupling the DOFs in the EOMs as much as possible. A base body is first prescribed in the multibody system and the EOMs of the system are projected into the coordinate systems relevant to this body. Only six basic EOMs of the system are required to capture 6 unknown DOFs of the base body because full use is made of the prescribed mechanical DOFs between contiguous bodies.

The 6×6 mass matrix is actually composed of two decoupled 3×3 mass matrices, one for translation and one for rotation. Each element within the matrix includes the inertial effects of all bodies. This condensation decreases the coupling between mass elements in the EOMs, and so minimizes the computational demand.

In prior work, Sweetman and Wang (Sweetman and Wang, 2012; Wang and Sweetman, 2012a,b) investigate the multibody dynamics of floating wind turbine systems with large-amplitude rotations using conservation of momentum. The MCM is an expansion of that prior work for application to a generalized multibody system connected by revolute or prismatic joints with prescribed relative motions between contiguous bodies. The method could be applied to imperfect joints only to the extent that relative motions between contiguous bodies can be prescribed; these constraint equations are enforced rigorously, precluding development of numerical constraint violations between bodies. This new work is focused on the formulation and solution of the six basic EOMs associated with the two 3×3 mass matrices. In addition to these basic EOMs, a generalized N-body system may also include the EOMs of the unknown DOFs of mechanically control joints (control equations) and EOMs describing kinematic relation between selected coordinates (constraint equations). In this case, the six basic EOMs include the effects of both the control and constraint equations through use of numerous system coordinates, including six reference point coordinates of the base body, and relative coordinates among contiguous bodies.

Derivation of the MCM first requires establishment of a set of proper coordinates. In the sections that follow, these coordinates are illustrated through a 2-body system and then generalized to a N-body system. The main derivation presented for the new method is general for a serial N-body system, such as a serial manipulator in robotics; in later sections it is then further generalized to more complicated forms with branches or loops. The new multi-body method is broadly applicable to systems of arbitrary topology, but application of topological algorithms applicable to arbitrary geometry are outside the scope of this work. The formulation can also be used on the inverse dynamic problem to calculate the internal forcing. Generally, the MCM can be applied to directly obtain internal forcing between any two contiguous bodies without needing to calculate the forcing at other joints. An example is presented in which the dynamics of a 6-body floating wind turbine system are simulated, in which the tower (including floater), nacelle, hub and the three blades are each represented as rigid bodies. Results of the forward and inverse dynamics are critically compared with the well-recognized NREL FAST aero-elastic simulator (Jonkman and Buhl, 2005).

2. Coordinate Systems and Dependent Coordinates

A set of proper coordinates is needed to unequivocally define the kinematics of a multibody system (Jalon and Bayo, 1994). Here, the necessary coordinate systems are first illustrated for a 2-body system, then the coordinates measured from such coordinate systems are demonstrated and generalized for an N-body system. Finally, the reasons and advantages for this choice of coordinates are described.

In the MCM, reference point coordinates are used to describe the motion of base body; relative coordinates are used to define the relative motions between contiguous bodies. Several proper coordinate systems are needed to define those coordinates. Fig. 1 shows the detailed definitions of various coordinate systems for a 2-body system. B_1 and B_2 are two rigid bodies connected by revolute joint J with known relative rotation. B_1 is specified as the base body. The angular rotation of B_2 relative to B_1 is always expressed relative to its initial position B_2^0. These relative rotations would be analogous to a moving tower (base body, B_1), on top of which is mounted a clock face and rotating hour hand (B_2). If the hour hand is initially at 12:00 (B_2^0), then all future positions of B_2 are measured relative to 12:00, regardless of motion of the tower. The coordinate systems (x_1, y_1, z_1) (the C_1 system) and (x_2, y_2, z_2) (the C_2 system) are body fixed and originate at the CM of bodies G_1 and G_2, respectively. The C_2^0 system originates at the CM of B_2^0 (G_2^0) and indicates the initial position of the C_2 system. The system-fixed coordinate system (x_s, y_s, z_s) (the C_s system) is located at the CM of the system (G_s), and is prescribed to be parallel to the C_1 system. The inertial coordinate system (X, Y, Z) (the C_I system) has its origin defined by the initial position of the CM of the base body, such that the radius vector from O to G_1 indicates the location of G_1 relative to its initial position.

The dependent coordinates for an N-body system are defined using similar coordinate systems. The reference point coordinates describe the translation and rotation of the base body measured from the inertial coordinate systems. The coordinates (X_1, X_2, X_3) measured from the C_I system are used to define the absolute motion of the CM of the C_s system, G_s. These coordinates are further transferred to the translations at the CM of the C_1 system, G_1, which are indicated by coordinates (X_{1b}, X_{2b}, X_{3b}) and define the 3 translational DOFs of the base body. The large-amplitude rotation of the base body w.r.t. the C_I system is described by the 1-2-3 sequenced Euler angles X_4-X_5-X_6 in Fig. 2, which are the 3 rotational DOFs of the base body. The (x', y', z') coordinate system translates with respect to the (X, Y, Z) system, with the origin always located at the CM of B_1. The C_1 system can be transformed from the (x', y', z') by: first rotating the upright tower about the x'-axis by angle X_4, then rotating about the resulting second coordinate axis through an angle X_5, and finally, rotating the tower about the z_t-axis through the third Euler angle, X_6. Thus, the reference point coordinates of the N-body system are represented by (X_{1b}, X_{2b}, X_{3b}) and (X_4, X_5, X_6). The relative coordinates of the N-body system can be measured by the rotation of C_i relative to the C_i^0 system and denoted by the angular velocity vector $\vec{\omega}_{B_i}^{C_i^0}$, which is the rotation of C_i relative to the C_i^0 system. Here, the relative coordinates include only one prescribed relative angular velocity to simplify the formulation of the EOMs. In general, more unknown relative coordinates can be applied, such as unknown relative rotation angles and orientations of the rotational axes, and can be interconnected through the control and constraint equations.

The reference point coordinates includes two unconventional features. First, they are used to describe the motion of an arbitrarily selected based body: the absolute translation at its CM and the absolute rotation as described by Euler angles. Second, only three rotational reference point coordinates (Euler angles) ultimately appear in the basic EOMs. Relative coordinates define the motion of each successive body (B_i) relative to its neighbor (B_{i-1}) along the kinematic chain, such that a cascading procedure can be applied to determine the absolute position of each body along the chain.

Generally, the dependency of all coordinates are represented by the combination of six basic EOMs, plus the control and constraint equations. In the derivation here, all relative coordinates at different joints are mechanically controlled, which is equivalent to an explicit solution to the control EOMs. The constraint equations are minimized in the derivation by prescribing certain relative coordinates equal to zero and simplifying the associated transformation matrices. However, the MCM can easily be expanded to multibody systems requiring all three kinds of EOMs. In general, the number of the basic and control EOMs is equal to the number of unknown DOFs of the system; the number of the constraint equations is equal to the difference between the number of dependent coordinates and that of the DOFs of the system. The control equations can be of any complexity, e.g. from the response of a spring to a highly complex numerical control system. The solution to the control equations at a single time-point effectively imposes a constraint at that time. In this sense, the control and constraint equations receive comparable treatment in the MCM: the solutions to them are represented in six basic EOMs in the form of various transforms.

An ideal choice of the coordinates both simplifies formulation of the EOMs and increases the efficiency of the numerical integration. Any coordinate introduced to a multibody dynamics problem becomes an unknown, which requires an additional equation to solve. The combination of reference point and relative coordinates guarantees that some dependent coordinates can be eliminated from the basic EOMs with only trivial computational demand. The basic coupled differential EOMs of the system are by far the most computationally demanding to solve, so minimizing the number of coordinates in these equations is of primary importance, e.g. (Nikravesh, 1990). Here, a very large number of coordinates is applied to facilitate the establishment of the EOMs, but the formulation enables elimination of all but six reference point coordinates from the basic EOMs.

3. Translational Equations of Motion

The conservation of linear momentum (Newton's second Law) is applied to the N-body system directly to avoid the calculation of internal forces between rigid bodies:

$$\sum_{i=1}^{N} \vec{F}_{B_i}^{C_I} = \left(\sum_{i=1}^{N} m_i\right) \vec{a}_{G_s}^{C_I} \tag{1}$$

where the forces $\sum_{i=1}^{N} \vec{F}_{B_i}^{C_I}$ are externally applied and projected into the inertial C_I system; $\sum_{i=1}^{N} m_i$ is the mass of the entire system; $\vec{a}_{G_s}^{C_I}$ is the linear acceleration of the CM of the system, $\vec{a}_{G_s}^{C_I} = [\ddot{X}_1, \ddot{X}_2, \ddot{X}_3]$. The solution to this translational EOMs is the motion of the CM of the system (G_s) measured from the (X, Y, Z) coordinate system. Direct solution of Eq. (1) for the motion of G_s enables decoupling of translations from rotations, but also frees G_s from being constrained to any body within the system.

The absolute position of G_s is used to compute a radius vector from the origin of inertial coordinate system (X, Y, Z) to each rigid body:

$$\vec{\rho}_{G_i/O}^{C_I} = \vec{\rho}_{G_s/O}^{C_I} + T_{C_s \to C_I} \vec{\rho}_{G_i/G_s}^{C_s} \quad (i \geqslant 1) \tag{2}$$

in which the vector $\vec{\rho}_{G_s/O}^{C_I}$ is the absolute motion of G_s in the C_I system: $\vec{\rho}_{G_s/O}^{C_I} = [X_1, X_2, X_3]$ and results from the integration of Eq. (1); radius vector $\vec{\rho}_{G_i/G_s}^{C_s}$ is from G_s to the CM of each rigid body, G_i, and is measured in the C_s system. These radius vectors depend on relative rotation between contiguous bodies and will be discussed later. The resulting $\vec{\rho}_{G_1/O}^{C_I}$ has special significance in that it describes the position of the base body using 3 translational reference point coordinates: $\vec{\rho}_{G_1/O}^{C_I} = [X_{1b}, X_{2b}, X_{3b}]$.

The transformation matrix from (x_s, y_s, z_s) to (X, Y, Z) needed in Eq. (2) can be expressed as (Sweetman and Wang, 2011):

$$T_{C_s \to C_I} = T_x(X_4) T_y(X_5) T_z(X_6) = \begin{bmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{bmatrix} \tag{3}$$

in which

$$\begin{aligned}
t_{11} &= \cos X_5 \cos X_6 \\
t_{12} &= -\cos X_5 \sin X_6 \\
t_{13} &= \sin X_5 \\
t_{21} &= \cos X_4 \sin X_6 + \cos X_6 \sin X_4 \sin X_5 \\
t_{22} &= \cos X_4 \cos X_6 - \sin X_4 \sin X_5 \sin X_6 \\
t_{23} &= -\cos X_5 \sin X_4 \\
t_{31} &= \sin X_4 \sin X_6 - \cos X_4 \cos X_6 \sin X_5 \\
t_{32} &= \cos X_6 \sin X_4 + \cos X_4 \sin X_5 \sin X_6 \\
t_{33} &= \cos X_4 \cos X_5
\end{aligned}$$

where $T_x(X_4)$, $T_y(X_5)$ and $T_z(X_6)$ are element transformation matrices (Abkowitz, 1969) that depends on rotational reference point coordinates (X_4, X_5, X_6).

Integration of Eq. (1) in a numerical time-domain simulation tool requires the initial displacement of G_s. This initial condition is computed from the prescribed initial displacements of the base body and initial joint displacements by solving Eq. (2) for $\vec{\rho}^{C_I}_{G_s/O}$. Eq. (2) is applied in future time steps to transform the motion of G_s resulting from integration of Eq. (1) to G_i for the calculation of external forces and moments. Thus, Eq. (2) can be used as both preprocessing and postprocessing procedures in the new method.

4. Rotational Equations of Motion

Beginning with conservation of angular momentum, the sum of the moments resulting from externally applied forces about the CM of a system of particles in the translating-rotating system, (x_s, y_s, z_s), equals the change of amplitude of the momentum within the coordinate system plus the change of direction of the momentum with respect to global coordinate system (e.g. (Hibbeler, 2004)). The rotational EOMs can be shown to be (Wang and Sweetman, 2012a):

$$\sum_{i=1}^{N} {}^{B_i}\vec{M}^{C_s}_{G_s} = \left({}^{s}\dot{\vec{H}}^{C_s}_{G_s}\right)_{C_s} + \vec{\omega}_{C_s} \times {}^{s}\vec{H}^{C_s}_{G_s} \tag{4}$$

where the notation $(\dot{\ })_{C_s}$ is the local derivative within the C_s system, i.e. the rate of change of the total quantity with respect to time. The external moments on each body, ${}^{B_i}\vec{M}^{C_s}_{G_s}$, are calculated about G_s and projected into the system-fixed C_s coordinate system; ${}^{s}\vec{H}^{C_s}_{G_s}$ is the total angular momentum of the N-body system, which is also calculated about G_s and projected into the C_s system; the absolute angular velocity of the C_s system, $\vec{\omega}_{C_s}$, describes the angular velocity of the C_s system with respect to the inertial coordinate system C_I, and can be shown to be (Wang and Sweetman, 2012a):

$$\vec{\omega}_{C_s} = \vec{\omega}^{C_1}_{B_1} = \begin{bmatrix} \dot{X}_4 \cos X_5 \cos X_6 + \dot{X}_5 \sin X_6 \\ -\dot{X}_4 \cos X_5 \sin X_6 + \dot{X}_5 \cos X_6 \\ \dot{X}_4 \sin X_5 + \dot{X}_6 \end{bmatrix} \tag{5}$$

which are Euler kinematic equations (Abkowitz, 1969) associated with the rotational reference point coordinates (X_4, X_5, X_6) and are equal to the absolute angular velocity of the base body decomposed to the C_1 system, $\vec{\omega}^{C_1}_{B_1}$, because the C_s system is parallel to the C_1 system.

Application of the theorem of conservation of angular momentum to a system requires that the momentum be calculated about the CM of the system. Here, the momentum of each body is computed about G_s, and projected onto a coordinate system parallel to the C_1 system. Computation of the absolute derivative of the angular momentum of each rigid body is simplified by decomposing its absolute angular velocity onto a body-fixed coordinate system (C_i), then projecting the resulting momentum back into the system-fixed coordinate system (C_s) in accordance with conventional Euler dynamics equations (e.g. (Hibbeler, 2004)). Using this method, the angular momentum of each body is computed in its own body-fixed coordinate system, and then transferred to the C_s system without introducing any new coordinates. Conservation of angular momentum for an N-body system requires development of general expressions of total angular momentum of the system and its local derivative.

4.1. Calculation of Angular Momentum

The total angular momentum with respect to G_s and projected to the C_s system can be summed over the N-body system:

$$ {}^{s}\vec{H}_{G_s}^{C_s} = \sum_{i=1}^{N} {}^{B_i}\vec{H}_{G_s}^{C_s} \tag{6} $$

in which the angular momentum of the i-th body about G_s and projected into the C_s system (${}^{B_i}\vec{H}_{G_s}^{C_s}$) is first calculated in the body-fixed coordinate system C_i about G_i, then transformed to the unified C_s system by transformation matrixes $T_{C_i \to C_s}$ and finally transferred to the unified reference point, G_s (Wang and Sweetman, 2012a):

$$ {}^{B_i}\vec{H}_{G_s}^{C_s} = T_{C_i \to C_s}\, {}^{B_i}\vec{H}_{G_i}^{C_i} + \vec{\rho}_{G_i/G_s}^{C_s} \times m_i \vec{v}_{G_i/G_s}^{C_s} \tag{7} $$

Each term in Eq. (7) is calculated individually, and then substituted back to Eq. (6) to obtain the total angular momentum.

The first term, ${}^{B_i}\vec{H}_{G_i}^{C_i}$, is the angular momentum of B_i calculated about the CM of the body, G_i, and decomposed onto C_i, the body-fixed coordinate system of B_i. This angular momentum of a rigid body is:

$$ {}^{B_i}\vec{H}_{G_i}^{C_i} = I_{B_i} \vec{\omega}_{B_i}^{C_i} \tag{8} $$

in which I_{B_i} is the tensor of moment of inertia of B_i. The absolute angular velocity of bodies numbered sequentially outward from the base body can be computed in a cascading format:

$$ \vec{\omega}_{B_i}^{C_i} = T_{C_{i-1} \to C_i} \vec{\omega}_{B_{i-1}}^{C_{i-1}} + T_{C_i^0 \to C_i} \vec{\omega}_{B_i}^{C_i^0} \quad (i \geqslant 2) \tag{9} $$

in which $\vec{\omega}_{B_i}^{C_i}$ is obtained along the kinematic chain by combining the effects of $\vec{\omega}_{B_{i-1}}^{C_{i-1}}$ and $\vec{\omega}_{B_i}^{C_i^0}$ and transforming the results into the common C_i system. The prescribed angular velocity of B_i relative to B_{i-1}, $\vec{\omega}_{B_i}^{C_i^0}$, is measured relative to the C_i^0 system, which is fixed to B_{i-1} and parallel to the initial position of B_i. The C_i^0 system is used to reference relative rotation between B_i and B_{i-1}. The transformation matrix $T_{C_{i-1} \to C_i}$ can be obtained as $T_{C_{i-1} \to C_i} = T_{C_i^0 \to C_i} T_{C_{i-1} \to C_i^0}$, in which $T_{C_{i-1} \to C_i^0}$ is time invariant and results from the initial direction cosine matrix between B_{i-1} and B_i^0; $T_{C_i^0 \to C_i}$ is time-dependent and can be calculated from known mechanical rotations at the joint.

Back to Eq. (7), the transformation matrix for any body motion relative to the base body can be expressed by the consecutive multiples of the transformation matrices between contiguous bodies along the kinematic chain:

$$T_{C_i \to C_s} = T_{C_i \to C_1} = \prod_{j=2}^{i} T_{C_j \to C_{j-1}} = \prod_{j=2}^{i} T_{C_j^0 \to C_{j-1}} T_{C_j \to C_j^0} \quad (i \geqslant 2) \tag{10}$$

The cross product terms in Eq. (7) transfer the reference point of angular momentum from the CM of the body to the CM of the system (e.g. (Hibbeler, 2004)). The CM of the system, G_s, is time-varying and not constrained to any rigid body, as dictated by arbitrary relative motion between rigid bodies (Wang and Sweetman, 2012a). Calculation of the radius vector ($\vec{\rho}^{C_s}_{G_i/G_s}$) can be illustrated through the 2-body model in Fig. 3. The CM of each body (G_1 and G_2) and of the system (G_s) are expressed through defining all relative motions due to joint rotations in the body-fixed C_1 system, which causes the radius vectors ($\vec{\rho}^{C_s}_{G_1/G_s}$ and $\vec{\rho}^{C_s}_{G_2/G_s}$) to be independent of the absolute motion of G_s. This independence makes the inertial forcing in three basic rotational EOMs independent of (X_1, X_2, X_3), which dramatically simplifies the formulation and solution of the basic EOMs.

Motions between bodies can be computed directly by using a transformation matrix to represent joint rotations; the CM of each body is represented in the C_1 system; the CM of the system, G_s, is computed as a weighted average; and the relative positions of G_1 and G_2 to G_s are obtained by vectorial combination (Fig. 3). Application of this procedure to an N-body system yields relative radius vector ($\vec{\rho}^{C_s}_{G_i/G_s}$):

$$\begin{aligned}
\vec{\rho}^{C_{i-1}}_{G_i/G_{i-1}} &= \vec{\rho}^{C_{i-1}}_{J_{i-1}/G_{i-1}} + T_{C_i \to C_{i-1}} \vec{\rho}^{C_i}_{G_i/J_{i-1}} \quad (i \geqslant 2) \\
\vec{\rho}^{C_1}_{G_i/G_1} &= \vec{\rho}^{C_1}_{G_{i-1}/G_1} + T_{C_{i-1} \to C_1} \vec{\rho}^{C_{i-1}}_{G_i/G_{i-1}} \quad (i \geqslant 3) \\
\vec{\rho}^{C_1}_{G_s/G_1} &= \frac{\sum_{i=1}^{N} m_i \vec{\rho}^{C_1}_{G_i/G_1}}{\sum_{i=1}^{N} m_i} \\
\vec{\rho}^{C_s}_{G_i/G_s} &= \vec{\rho}^{C_1}_{G_i/G_s} = \vec{\rho}^{C_1}_{G_i/G_1} - \vec{\rho}^{C_1}_{G_s/G_1} \quad (i \geqslant 1)
\end{aligned} \tag{11}$$

in which radius vectors $\vec{\rho}^{C_{i-1}}_{J_{i-1}/G_{i-1}}$ and $\vec{\rho}^{C_i}_{G_i/J_{i-1}}$ indicate the fixed locations of the joint connecting two contiguous rigid bodies. The final radius vector in the C_s system, $\vec{\rho}^{C_s}_{G_i/G_s}$, is equal to $\vec{\rho}^{C_1}_{G_i/G_s}$ in the C_1 system.

The relative linear velocity in Eq. (7), $\vec{v}^{C_s}_{G_i/G_s}$, is computed as the absolute derivative of the radius vector of $\vec{\rho}^{C_s}_{G_i/G_s}$:

$$\vec{v}^{C_s}_{G_i/G_s} = \frac{d\vec{\rho}^{C_s}_{G_i/G_s}}{dt} = \left(\dot{\vec{\rho}}^{C_s}_{G_i/G_s}\right)_{C_s} + \vec{\omega}_{C_s} \times \vec{\rho}^{C_s}_{G_i/G_s} \quad (i \geqslant 1) \tag{12}$$

the form of which is similar to Eq. (4) because $\vec{\rho}^{C_s}_{G_i/G_s}$ is decomposed into the rotating C_s system. The translation of the system, (X_1, X_2, X_3), does not appear in Eqs. (11) and (12), indicating that motions due to rotation are independent of translation of the system, which greatly facilitates numerical integration.

Eqs. (6)-(12) can be condensed into a single expression for the total angular momentum of the N-body system about the reference point G_s:

$$
\begin{aligned}
{}^{s}\vec{H}^{C_s}_{G_s} &= \sum_{i=1}^{N}\left(T_{C_i\to C_s}\,{}^{B_i}\vec{H}^{C_i}_{G_i} + \vec{\rho}^{C_s}_{G_i/G_s}\times m_i\vec{v}^{C_s}_{G_i/G_s}\right) \\
&= P_1\vec{\omega}^{C_1}_{B_1} + Q_1 + P_2\vec{\omega}^{C_1}_{B_1} + Q_2 \qquad (13)
\end{aligned}
$$

in which the terms $P_1\vec{\omega}^{C_1}_{B_1}$ and Q_1 represent the total angular momentum of each body calculated about its own CM (G_i), but projected into the unified C_s system, i.e. $\sum_{i=1}^{N} T_{C_i\to C_s}\,{}^{B_i}\vec{H}^{C_i}_{G_i}$; the terms $P_2\vec{\omega}^{C_1}_{B_1}$ and Q_2 represent the effect of transferring from the CM of each body (G_i) to the unified reference point(G_s), i.e. $\sum_{i=1}^{N}\vec{\rho}^{C_s}_{G_i/G_s}\times m_i\vec{v}^{C_s}_{G_i/G_s}$. Each of the four coefficients represents a sum over the entire system:

$$
\begin{aligned}
P_1 &= \sum_{i=1}^{N} T_{C_i\to C_s} I_{B_i} T_{C_1\to C_i} \\
Q_1 &= \sum_{i=2}^{N}\left[\left(\sum_{j=i}^{N} T_{C_j\to C_s} I_{B_j} T_{C_i^0\to C_j}\right)\vec{\omega}^{C_i^0}_{B_i}\right] \\
P_2 &= \sum_{i=1}^{N} m_i\bar{\rho}_{G_i/G_s} \\
Q_2 &= \sum_{i=1}^{N} m_i\vec{\rho}^{C_s}_{G_i/G_s}\times\left(\dot{\vec{\rho}}^{C_s}_{G_i/G_s}\right)_{C_s} \qquad (14)
\end{aligned}
$$

where $\bar{\rho}_{G_i/G_s}$ is introduced as a computational convenience. The calculation of $P_2\vec{\omega}^{C_1}_{B_1}$ is simplified through use of a matrix identity $\vec{\rho}^{C_s}_{G_i/G_s}\times(\vec{\omega}^{C_1}_{B_1}\times\vec{\rho}^{C_s}_{G_i/G_s}) = \bar{\rho}_{G_i/G_s}\vec{\omega}^{C_1}_{B_1}$ (Stoneking, 2007), where:

$$
\bar{\rho}_{G_i/G_s} = \begin{bmatrix} \rho_2^2+\rho_3^2 & -\rho_1\rho_2 & -\rho_1\rho_3 \\ -\rho_1\rho_2 & \rho_1^2+\rho_3^2 & -\rho_2\rho_3 \\ -\rho_1\rho_3 & -\rho_2\rho_3 & \rho_1^2+\rho_2^2 \end{bmatrix} \qquad (15)
$$

in which the three elements of any radius vector $\vec{\rho}^{C_s}_{G_i/G_s}$ are represented by $[\rho_1, \rho_2, \rho_3]$. This matrix identity enables extraction of the angular velocity ($\vec{\omega}^{C_1}_{B_1}$) from the double cross product term, which facilitates the establishment of first-order decoupled EOMs and greatly simplifies derivative calculations.

4.2. Calculation of Local Derivative of Angular Momentum

The rotational EOMs (Eq. (4)) require the local derivative of the total angular momentum within the C_s system, which is calculated from Eq. (13):

$$
\left({}^{s}\dot{\vec{H}}^{C_s}_{G_s}\right)_{C_s} = (P_1 \dot{+} P_2)_{C_s}\vec{\omega}^{C_1}_{B_1} + (P_1+P_2)\left(\dot{\vec{\omega}}^{C_1}_{B_1}\right)_{C_s} + (Q_1 \dot{+} Q_2)_{C_s} \qquad (16)
$$

in which $(P_1 \dot{+} P_2)_{C_s} = (\dot{P}_1)_{C_s} + (\dot{P}_2)_{C_s}$, $(Q_1 \dot{+} Q_2)_{C_s} = (\dot{Q}_1)_{C_s} + (\dot{Q}_2)_{C_s}$. Eq. (16) includes the local derivatives of the four coefficients in Eq. (14) and the angular velocity of the base body in Eq. (5), which are calculated consecutively.

First, taking the derivative of Eq. (14) yields:

$$
\begin{aligned}
(\dot{P}_1)_{C_s} &= \sum_{i=1}^{N} \dot{T}_{C_i\to C_s} I_{B_i} T_{C_1\to C_i} + T_{C_i\to C_s} I_{B_i} \dot{T}_{C_1\to C_i} \\
(\dot{Q}_1)_{C_s} &= \sum_{i=2}^{N} \left\{ \left[\sum_{j=i}^{N} \left(\dot{T}_{C_j\to C_s} I_{B_j} T_{C_i^0\to C_j} + T_{C_j\to C_s} I_{B_j} \dot{T}_{C_i^0\to C_j} \right) \right] \vec{\omega}_{B_i}^{C_i^0} \right. \\
&\quad \left. + \left(\sum_{j=i}^{N} T_{C_j\to C_s} I_{B_j} T_{C_i^0\to C_j} \right) \dot{\vec{\omega}}_{B_i}^{C_i^0} \right\} \\
(\dot{P}_2)_{C_s} &= \sum_{i=1}^{N} m_i \dot{\tilde{\rho}}_{G_i/G_s} \\
(\dot{Q}_2)_{C_s} &= \sum_{i=1}^{N} m_i \vec{\rho}_{G_i/G_s}^{C_s} \times \left(\ddot{\vec{\rho}}_{G_i/G_s}^{C_s} \right)_{C_s}
\end{aligned} \tag{17}
$$

in which the derivatives of the transformation matrices can be obtained by a cascading method. For example, the matrix derivative $\dot{T}_{C_1\to C_i}$ can be expressed as:

$$
\dot{T}_{C_1\to C_i} = \dot{T}_{C_{i-1}\to C_i} T_{C_1\to C_{i-1}} + T_{C_{i-1}\to C_i} \dot{T}_{C_1\to C_{i-1}} \quad (i \geqslant 2) \tag{18}
$$

The derivative of transformation matrix $T_{C_{i-1}\to C_i}$ can be obtained by $\dot{T}_{C_{i-1}\to C_i} = \dot{T}_{C_i^0\to C_i} T_{C_{i-1}\to C_i^0}$ by considering that $\dot{T}_{C_{i-1}\to C_i^0} = 0$, since the direction cosine matrix $T_{C_{i-1}\to C_i^0}$ is time independent.

The matrix derivative $\dot{\tilde{\rho}}_{G_i/G_s}$ in Eq. (17) can be obtained by:

$$
\dot{\tilde{\rho}}_{G_i/G_s} = \begin{bmatrix} 2(\rho_2\dot{\rho}_2 + \rho_3\dot{\rho}_3) & -\dot{\rho}_1\rho_2 - \rho_1\dot{\rho}_2 & -\dot{\rho}_1\rho_3 - \rho_1\dot{\rho}_3 \\ -\dot{\rho}_1\rho_2 - \rho_1\dot{\rho}_2 & 2(\rho_1\dot{\rho}_1 + \rho_3\dot{\rho}_3) & -\dot{\rho}_2\rho_3 - \rho_2\dot{\rho}_3 \\ -\dot{\rho}_1\rho_3 - \rho_1\dot{\rho}_3 & -\dot{\rho}_2\rho_3 - \rho_2\dot{\rho}_3 & 2(\rho_1\dot{\rho}_1 + \rho_2\dot{\rho}_2) \end{bmatrix} \tag{19}
$$

where the local derivative $\left(\dot{\vec{\rho}}_{G_i/G_s}^{C_s}\right)_{C_s}$ is defined as $\left(\dot{\vec{\rho}}_{G_i/G_s}^{C_s}\right)_{C_s} = [\dot{\rho}_1, \dot{\rho}_2, \dot{\rho}_3]$ and can be obtained by taking the derivative of Eq. (11):

$$
\begin{aligned}
\left(\dot{\vec{\rho}}_{G_i/G_{i-1}}^{C_{i-1}}\right)_{C_{i-1}} &= \dot{T}_{C_i\to C_{i-1}} \vec{\rho}_{G_i/J_{i-1}}^{C_i} \quad (i \geqslant 2) \\
\left(\dot{\vec{\rho}}_{G_i/G_1}^{C_1}\right)_{C_1} &= \left(\dot{\vec{\rho}}_{G_{i-1}/G_1}^{C_1}\right)_{C_1} + \dot{T}_{C_{i-1}\to C_1} \vec{\rho}_{G_i/G_{i-1}}^{C_{i-1}} + T_{C_{i-1}\to C_1} \left(\dot{\vec{\rho}}_{G_i/G_{i-1}}^{C_{i-1}}\right)_{C_{i-1}} \quad (i \geqslant 3) \\
\left(\dot{\vec{\rho}}_{G_s/G_1}^{C_1}\right)_{C_1} &= \frac{\sum_{i=1}^{N} m_i \left(\dot{\vec{\rho}}_{G_i/G_1}^{C_1}\right)_{C_1}}{\sum_{i=1}^{N} m_i} \\
\left(\dot{\vec{\rho}}_{G_i/G_s}^{C_s}\right)_{C_s} &= \left(\dot{\vec{\rho}}_{G_i/G_1}^{C_1}\right)_{C_1} - \left(\dot{\vec{\rho}}_{G_s/G_1}^{C_1}\right)_{C_1} \quad (i \geqslant 1)
\end{aligned} \tag{20}
$$

in which $\vec{\rho}_{G_i/J_{i-1}}^{C_i}$ in the first expression is time-independent. Similarly, the second order derivative in Eq. (17), $\left(\ddot{\vec{\rho}}_{G_i/G_s}^{C_s}\right)_{C_s}$, can be obtained by taking the derivative of Eq. (20).

Back to Eq. (16), the local derivative of the angular velocity of the base body can be found by taking the derivative of Eq. (5):

$$\left(\dot{\vec{\omega}}^{C_1}_{B_1}\right)_{C_s} = T_\omega \begin{bmatrix} \ddot{X}_4 \\ \ddot{X}_5 \\ \ddot{X}_6 \end{bmatrix} + RE \tag{21}$$

where the angular acceleration vector is expressed in the general matrix form. Extracting the vector $[\ddot{X}_4\ \ddot{X}_5\ \ddot{X}_6]$, enables explicit expression of these three rotational reference point coordinates in a form convenient for numerical simulation. The vector RE and the matrix T_ω can be expressed by:

$$RE = \begin{bmatrix} -\dot{X}_4\dot{X}_5 \sin X_5 \cos X_6 - \dot{X}_4\dot{X}_6 \cos X_5 \sin X_6 + \dot{X}_5\dot{X}_6 \cos X_6 \\ \dot{X}_4\dot{X}_5 \sin X_5 \sin X_6 - \dot{X}_4\dot{X}_6 \cos X_5 \cos X_6 - \dot{X}_5\dot{X}_6 \sin X_6 \\ \dot{X}_4\dot{X}_5 \cos X_5 \end{bmatrix}$$

$$T_\omega = \begin{bmatrix} \cos X_5 \cos X_6 & \sin X_6 & 0 \\ -\cos X_5 \sin X_6 & \cos X_6 & 0 \\ -\sin X_5 & 0 & 1 \end{bmatrix} \tag{22}$$

which is singular if $\cos X_5 = 0$.

5. Application

The derivation results for the total angular momentum and its derivative in Section 4 can be directly applied to formulate the rotational EOMs of a serial N-body system, which is combined with the translational EOMs to form six basic EOMs of the system. A numerical implementation of the new method is also presented.

Eq. (13) can be applied to express the total angular momentum of the N-body system required in the rotational EOMs (Eq. (4)). The first two terms, $P_1\vec{\omega}^{C_1}_{B_1}$ and Q_1, represent the angular momentum of the entire system projected into the unified C_s system but calculated about the CM of individual bodies. $P_1\vec{\omega}^{C_1}_{B_1}$ represents the instantaneous angular momentum of the N-body system rotating with the base body. The coefficient P_1 can be generalized by first transferring the angular velocity of the base body ($\vec{\omega}^{C_1}_{B_1}$) to each rigid body through the transformation matrix ($T_{C_1 \to C_i}$), then multiplying the result with the corresponding inertia tensor (I_{B_i}) and finally transforming the angular momentum back to the unified coordinate system (C_s) and summing up. The term Q_1 represents the contribution of relative rotation among the bodies ($\vec{\omega}^{C^0_i}_{B_i}$) to the total angular momentum. The angular velocity at each joint ($\vec{\omega}^{C^0_i}_{B_i}$) is transferred to all of the bodies affected by this joint along the kinematic chain, then multiplied by the corresponding inertia tensor and finally transformed back to the unified C_s system.

The effect of transferring the reference points is represented in $P_2\vec{\omega}^{C_1}_{B_1}$ and Q_2, in which radius vector $\vec{\rho}^{C_s}_{G_i/G_s}$ and matrix $\bar{\rho}_{G_i/G_s}$ represent the effect of the unconstrained CM of the system. Computing this effect requires formulation of the radius vectors from the CM of the system to the CM of each body. First, the kinematic chain is decomposed into individual links similar to Fig. 3 to apply Eq. (11). Each of the links is made of two contiguous bodies (B_{i-1} and B_i) connected by a mechanical joint (J_{i-1}). The radius vector from G_{i-1} to G_i, $\vec{\rho}^{C_{i-1}}_{G_i/G_{i-1}}$, is projected into the C_{i-1} system and calculated for each link. The fixed radius vectors between each joint and the two connected bodies, $\vec{\rho}^{C_{i-1}}_{J_{i-1}/G_{i-1}}$ and $\vec{\rho}^{C_i}_{G_i/J_{i-1}}$, are time-independent if expressed in body-fixed coordinate systems C_{i-1} and C_i. Second, the cascading expressions in the first two lines of Eq. (11) are applied repeatedly to calculate the radius vector, $\vec{\rho}^{C_1}_{G_i/G_1}$, from the CM of the base body to that of each body along the chain. Third, the CM of the system (G_s) within the C_1 system is calculated as a weighted average. Finally, the radius vectors needed in the coefficients P_2 and Q_2, $\vec{\rho}^{C_s}_{G_i/G_s}$ and $\vec{v}^{C_s}_{G_i/G_s}$, are obtained by vector combination and absolute derivative calculation, respectively. The local derivative of the relative radius vector $\left(\dot{\vec{\rho}}^{C_s}_{G_i/G_s}\right)_{C_s}$ needed in the velocity expression of Eq. (12) is obtained by calculating the derivative of each element within $\vec{\rho}^{C_s}_{G_i/G_s}$. Eq. (15) is used to update $\bar{\rho}_{G_i/G_s}$ based on $\vec{\rho}^{C_s}_{G_i/G_s}$.

Generally, the local derivative of the total angular momentum in Eq. (16) is computed using Eqs. (17) and (21). The angular acceleration of the base body is decoupled from the translational acceleration, enabling explicit expression of $[\ddot{X}_4, \ddot{X}_5, \ddot{X}_6]$ independent of $[\ddot{X}_1, \ddot{X}_2, \ddot{X}_3]$.

Finally, Eqs. (1) and (4) are rearranged to explicitly express the unknown accelerations:

$$\begin{aligned} \sum_{i=1}^{N} m_i \begin{bmatrix} \ddot{X}_1 \\ \ddot{X}_2 \\ \ddot{X}_3 \end{bmatrix} &= \sum_{i=1}^{N} \vec{F}^{C_I}_{B_i} \\ (P_1 + P_2)T_\omega \begin{bmatrix} \ddot{X}_4 \\ \ddot{X}_5 \\ \ddot{X}_6 \end{bmatrix} &= \sum_{i=1}^{N} {}^{B_i}\vec{M}^{C_s}_{G_s} - (P_1 \dot{+} P_2)_{C_s} \vec{\omega}^{C_1}_{B_1} - (Q_1 \dot{+} Q_2)_{C_s} \\ &\quad -(P_1 + P_2)RE - \vec{\omega}_{C_s} \times {}^{s}\vec{H}^{C_s}_{G_s} \end{aligned} \tag{23}$$

Eq. (23) represents the six basic EOMs of the N-body system as a balance between inertial and external forcing. The terms on the LHS are inertial forcing, which depend on unknown translational and rotational accelerations; the term $(Q_1 \dot{+} Q_2)_{C_s}$ in the RHS of basic rotational EOMs depends on the accelerations of known relative coordinates. Other terms in the rotational EOMs are the velocity-dependent inertia terms (Jalon and Bayo, 1994), including $(P_1 \dot{+} P_2)_{C_s}\vec{\omega}^{C_1}_{B_1}$, $(P_1 + P_2)RE$ and $\vec{\omega}_{C_s} \times {}^{s}\vec{H}^{C_s}_{G_s}$. The overall result is that the influence of prescribed relative coordinates between contiguous bodies has been used throughout the formulation of the basic EOMs. The factor $\sum_{i=1}^{N} m_i$ is the total mass of the N-body system. It could be simply expanded to an equivalent diagonal mass matrix relevant to the absolute motion of G_s, (X_1, X_2, X_3); $(P_1 + P_2)T_\omega$ is the mass matrix associated with the rotational reference point coordinates of the base body, (X_4, X_5, X_6). These two mass matrices represent the combined effect of the mass and inertia of every body within the multibody system and are equivalent to collapsing the larger mass matrix of any one of the conventional methods. More importantly, these rotational and translational mass matrices are decoupled, enabling a significant numerical advantage over other analytical methods.

First-order decoupled EOMs convenient for use in numerical integration can be easily obtained from Eq. (23); the numerical values of the matrices and vectors in Eq. (23) are updated at every time step. The set of three basic translational EOMs can be integrated separately from the basic rotational EOMs. In this procedure, the kinematics of the translational and rotational reference point coordinates (including velocity and acceleration) are thoroughly decoupled in the inertial forcing. The coordinates (X_1, X_2, X_3) are connected with the translational reference point coordinates of the base body, (X_{1b}, X_{2b}, X_{3b}), and the positions of the CM of other bodies through pre- and postprocessing procedures based on Eq. (2). These procedures are used to prescribe initial conditions about G_1 at the beginning of the first time step and compute external forcing applied to each body in the future time steps.

6. Expansion for General N-Body Systems

The derivation for serial N-body systems can be applied to more general N-body systems with only minor expansion. These more complicated systems may include open-chains with branches and closed-chain loops of rigid bodies. The final six basic EOMs, Eqs. (1) and (4), remain unchanged, but some intermediate equations require additional explanation. The increased complexity of the joints requires additional relative coordinates between contiguous bodies, which may introduce additional control and constraint equations. Application to more complicated systems is first discussed, followed by applications including more complicated joints.

The previous derivation was for a serial kinematic chain which enabled sequential numbering, starting from the base body and progressing through all rigid bodies. However, a simple kinematic chain does not exist for an open-chain system with branches (a tree system) or for the combination of a tree system and a closed-chain system. The expansion of the new method only requires that each body be connected by some kinematic chain to the base body, and that each body be uniquely numbered. As in the serial case, radius vectors between the CM of each body and its contiguous joints, as well as transformation matrices between all pairs of contiguous bodies must be determined. These transformations can be used with any kinematic chain to find the position and orientation of any body relative to the base body. Two equations need to be re-examined for application to non-serial N-body systems: Eq. (11) and (14). The first and second expressions in Eq. (11) can be applied to any kinematic chain in a non-serial N-body system as long as the bodies and joints are uniquely numbered from 1 to N. The calculation of the relative position of G_s in the C_1 system ($\vec{\rho}^{C_1}_{G_s/G_1}$) in the third expression already considers the relative positions of all bodies within the N-body system, and can be used directly, as can the fourth and fifth expressions of $\vec{\rho}^{C_s}_{G_i/G_s}$ and $\vec{v}^{C_s}_{G_i/G_s}$. The only change required to Eq. (14) is to modify the internal summation in Q_1 to exclude bodies not affected by the relative angular velocity $\vec{\omega}^{C^0_i}_{B_i}$, i.e. to exclude bodies on other branches.

Just as the method can be applied to systems of any complexity, it can also be applied to joints of any complexity. Any joint can be represented by relative coordinates between contiguous bodies. As long as the relative motion described by these additional relative coordinates are prescribed, no additional EOMs need to be solved beyond six basic EOMs. For example, a cylindrical joint allowing both translation and rotation requires one additional relative coordinate to describe the prescribed translation of the joint hinge relative to its initial position. This motion is added to the constant vector $\vec{\rho}^{C_i}_{G_i/J_{i-1}}$ in Eq. (11).

Generally, if the relative coordinates at joints are unknown, the control and constraint equations governing these coordinates are fully coupled with six basic EOMs. The sum $(Q_1 \dot{+} Q_2)_{C_s}$ in Eq. (17) introduces accelerations of unknown relative coordinates into the basic EOMs. The control and constraint equations may also depend on accelerations of six reference point coordinates, in which case, the accelerations of relative coordinates in these equations can be expressed as a function of the accelerations of the reference point coordinates and substituted into $(Q_1 \dot{+} Q_2)_{C_s}$ in the basic EOMs to compute $(\ddot{X}_4, \ddot{X}_5, \ddot{X}_6)$. The translational accelerations of the base body, $(\ddot{X}_1, \ddot{X}_2, \ddot{X}_3)$, still result from the first equation in Eq. (23). Finally, explicit expressions of all the accelerations of the chosen coordinates are applied prior to the numerical integration.

7. Inverse Dynamics of an N-body system

The computational efficiency of the forward dynamic simulation method is enhanced by avoiding the need to calculate internal forces and moments between bodies. However, it is commonly necessary to quantify this internal forcing. Inverse dynamics can be used to calculate internal forces and moments between arbitrarily selected contiguous bodies using the kinematics resulting from a forward dynamic simulation.

The first step is to divide the N-body system into two subsystems based on the position of the desired unknown internal forcing. The division must be made such that these two subsystems are connected by either one or two common joints. Known kinematic time histories of both inertial and applied external forcing on each body are used for the calculation of the internal forcing by applying the conservation of momentum to one (for one-joint case) or two (for two-joint case) subsystems. The new method is first derived for the one-joint case and then simply expanded to the two-joint case.

Two subsystems (s_1 and s_2) in the original system (s) are connected by one common joint J. The conservation of linear momentum (Newton's second Law) can be applied to either of these two subsystems to determine forces between them. Subsystem s_1 includes N_1 bodies. Newton's second Law applied to subsystem s_1 is:

$$ {}^{s}\sum_{i=1}^{N_1} \vec{F}_{B_i}^{C_I} + \vec{F}_J^{C_I} \quad = \quad \left(\sum_{i=1}^{N_1} m_i\right) \vec{a}_{G_{s_1}}^{C_I} \tag{24} $$

The unknown force on joint J is internal for the original s system and external for the subsystem s_1. This force is denoted as $\vec{F}_J^{C_I}$ and equal to the difference between the external forces applied to each body in the s_1 system and ${}^{s}\sum_{i=1}^{N_1} \vec{F}_{B_i}^{C_I}$, the external forces on all N_1 bodies in the original s system. The vector ${}^{s}\sum_{i=1}^{N_1} \vec{F}_{B_i}^{C_I}$ is known from Eq. (1) in the forward dynamic simulation. The vector $\vec{a}_{G_{s_1}}^{C_I}$ in the RHS is the acceleration of G_{s_1} (the CM of the subsystem s_1) and can be expressed by $\vec{a}_{G_{s_1}}^{C_I} = \ddot{\vec{\rho}}_{G_{s_1}/O}^{C_I}$. Similar to the third expression in Eq. (11), the radius vector $\vec{\rho}_{G_{s_1}/O}^{C_I}$ can be shown to be: $\vec{\rho}_{G_{s_1}/O}^{C_I} = \frac{\sum_{i=1}^{N_1} m_i \vec{\rho}_{G_i/O}^{C_I}}{\sum_{i=1}^{N_1} m_i}$, in which the radius vectors $\vec{\rho}_{G_i/O}^{C_I}$ are known and result from Eq. (2) in the forward dynamic simulation. To summarize, the unknown internal forces ($\vec{F}_J^{C_I}$) can be obtained by: first saving the time histories of ${}^{s}\sum_{i=1}^{N_1} \vec{F}_{B_i}^{C_I}$ and $\vec{\rho}_{G_i/O}^{C_I}$ in forward dynamics; then using $\vec{\rho}_{G_i/O}^{C_I}$ to calculate the time history of $\vec{\rho}_{G_{s_1}/O}^{C_I}$; numerically calculating the second order derivative of $\vec{\rho}_{G_{s_1}/O}^{C_I}$ to obtain the acceleration of G_{s_1} as well as the RHS $\left(\sum_{i=1}^{N_1} m_i\right) \vec{a}_{G_{s_1}}^{C_I}$; finally subtracting the time history of ${}^{s}\sum_{i=1}^{N_1} \vec{F}_{B_i}^{C_I}$ from that of the RHS to obtain $\vec{F}_J^{C_I}$.

Moments between subsystems connected by a single joint are found by a similar procedure. The conservation of angular momentum (Newton's second Law) can be applied to the subsystem

s_1 to determine internal moments in the original s system:

$$ {}^{s}\sum_{i=1}^{N_1} {}^{B_i}\vec{M}^{C_s}_{G_{s_1}} + \vec{M}^{C_s}_{F_J} + \vec{M}^{C_s}_{J} \quad = \quad \left({}^{s_1}\dot{\vec{H}}^{C_s}_{G_{s_1}} \right)_{C_s} + \vec{\omega}_{C_s} \times {}^{s_1}\vec{H}^{C_s}_{G_{s_1}} \tag{25} $$

where the LHS includes both the force moments (moments resulting from a force) and couple moments (moments resulting from pairs of equal and opposite applied forces). The force moments are calculated about G_{s_1}, while the couple moments are free vectors and independent of the reference point. The paired forces do not appear in the balance of forces in Eq. (24), but their effects do appear in the balance of moments in Eq. (25). Here, ${}^{s}\sum_{i=1}^{N_1} {}^{B_i}\vec{M}^{C_s}_{G_{s_1}}$ represents the external force and couple moments on all N_1 bodies within the original s system, as applied in the original forward dynamic simulation. The term $\vec{M}^{C_s}_{F_J}$ represents the force moments applied on joint J and can be shown to be $\vec{\rho}^{C_s}_{J/G_{s_1}} \times (T_{C_I \to C_s} \vec{F}^{C_I}_{J})$, in which the internal force $\vec{F}^{C_I}_{J}$ is calculated using Eq. (24); the radius vector $\vec{\rho}^{C_s}_{J/G_{s_1}}$ can be obtained using: $\vec{\rho}^{C_s}_{J/G_{s_1}} = T_{C_{N_1} \to C_s} \vec{\rho}^{C_{N_1}}_{J/G_{N_1}} + \vec{\rho}^{C_s}_{G_{N_1}/G_{s_1}}$. The couple moments on the joint J ($\vec{M}^{C_s}_{J}$) are the desired internal moments (e.g. (Kurfess, 2004)). In the RHS, vector ${}^{s_1}\vec{H}^{C_s}_{G_{s_1}}$ is the total angular momentum of the N_1 bodies about the CM of subsystem s_1 (G_{s_1}): ${}^{s_1}\vec{H}^{C_s}_{G_{s_1}} = {}^{s_1}\vec{H}^{C_s}_{G_s} - \vec{\rho}^{C_s}_{G_s/G_{s_1}} \times m_{s_1} \vec{v}^{C_s}_{G_s/G_{s_1}}$. Here, the vector ${}^{s_1}\vec{H}^{C_s}_{G_s}$ is the total angular momentum of N_1 bodies about G_s; the vector $\vec{\rho}^{C_s}_{G_s/G_{s_1}}$ can be calculated as $\vec{\rho}^{C_s}_{G_s/G_{s_1}} = \vec{\rho}^{C_s}_{G_s/G_1} - \vec{\rho}^{C_s}_{G_{s_1}/G_1}$. Thus, the inertial forcing in the RHS of Eq. (25) is computed using kinematic results for all N_1 bodies resulting from the forward dynamic simulation. The unknown internal moments ($\vec{M}^{C_s}_{J}$) are obtained by subtracting ${}^{s}\sum_{i=1}^{N_1} {}^{B_i}\vec{M}^{C_s}_{G_{s_1}}$ and $\vec{M}^{C_s}_{F_J}$ from the time history of the inertial forcing, i.e. the RHS of Eq. (25).

A two-joint case is solved in a similar manner using Eqs. (24) and (25). Application of Eq. (24) to each of the two subsystems results in two sets of equations with two sets of unknown internal forces, which can be solved simultaneously. After obtaining the two internal forces, Eq. (25) can be applied to each of the two subsystems and solved simultaneously for the internal moments at the two joints.

Configurations for which the number of the desired unknown internal forces or moments is greater than 2 can be solved using multiple applications of this method by dividing the original s system into a series of one or two-joint cases. Calculation of internal forcing for overdetermined systems, e.g., those with redundant constraints that would constitute statically indeterminant structures, is beyond the scope of this work.

8. Example

The new multibody formulation (MCM) is applied to a 6-body compliant floating wind turbine design. The floating system is represented as a 3-body serial chain consisting of the tower (body 1), nacelle (body 2), hub (body 3) plus three blades (bodies 4-6) branching from the hub. The tower is being used as the base body. The compliant design being analyzed is obtained by truncating the spar cylinder of OC3-Hywind model (Jonkman, 2010). The truncation reduces structural weight and the available hydrostatic restoring moment, which allows larger pitch angles. Fig. 4 shows the coordinate systems used for the MCM, with only one of the three body-fixed coordinate systems on the blades is shown. The (X_M, Y_M, Z_M) system is defined to enable direct comparison of simulation results with those of FAST, in which the reference point is usually prescribed to be on the still water level. In this implementation, rotation X_5 is aligned with the pitch direction, which precludes complications associated with the singularity of Equation 22. The six unknown reference point coordinates in the basic EOMs are the three translational DOFs of the CM of the tower and three rotational DOFs of the base body. The known relative coordinates are yaw rate of the nacelle relative to the tower, spin rate of the hub relative to the nacelle, and blade-pitch rates relative to hub.

The topsides (nacelle, hub and blades) are the same as that in OC3-Hywind: the moment of inertia of nacelle about yaw axis is 2.61×10^6 kg·m^2; the moment of inertia of rotor about spin axis is 3.54×10^7 kg·m^2. The moments of inertia of the tower w.r.t. the (x_1, y_1, z_1) are 5.85×10^9 kg·m^2 and 1.12×10^8 kg·m^2 in tilt (roll and/or pitch) and yaw, respectively. The four taught-leg mooring lines are each assumed to be a straight axial spring with stiffness of 6.81×10^5 N/m and length of 564-m in a 320-m water depth location. The origin of the global coordinate system (X, Y, Z) is the initial position of the CM of the tower, 58.67 m below still water.

Application of the MCM requires that the 6-body model be decomposed into three serial kinematic chains. Each chain starts from the base body (tower) and terminates including one of the three blades. Along each chain, the angular velocity of each body is obtained using Eqs. (5) and (9). The common part of all chains (bodies 1, 2 and 3) needs to be derived only once. The transformation matrices along each chain are obtained using Eq. (10). The prescribed position of each joint relative to the CM of the contiguous body is then used to determine the kinematics of each body relative to the CM of the system using Eqs. (11) and (12). The angular momentum of the system and its local derivative can be written directly through use of Eqs. (13) and (16). The effects of the rotations are combined with translations using Eq. (2).

The results begin with verification of the MCM through comparison of results with those computed using the popular wind turbine dynamics software FAST (Jonkman, 2007). FAST uses Kane's method to formulate the EOMs of the wind turbine system. Under the free vibration case, both the global motion from forward dynamics and internal forcing from inverse dynamics are compared to simulation results from FAST.

8.1. Global motion from forward dynamics

Figs. 5–6 show time histories of global motion computed using FAST and those computed using the MCM for a small-amplitude free vibration case. The rotational DOFs of the tower are transferred to the inertial coordinate system used in FAST to enable direct comparison between (X_4, X_5, X_6) and pitch, roll and yaw, which is valid for small-amplitude rotation (Abkowitz, 1969). The translational DOFs, (X_{1b}, X_{2b}, X_{3b}), are transferred to the waterplane to enable direct comparison with the sway, surge and heave computed in FAST, which are measured from the (X_M, Y_M, Z_M) system in Fig. 4. Constant nacelle yaw (1.2 deg/sec), hub spin (12.1 rpm) and blade-pitch rate (1.2 deg/sec) are prescribed during the simulation. Here, both hydrodynamics and aerodynamics have been disabled in FAST. The only external forces acting on the base body are from the mooring lines and buoyancy, both of which are represented in the user-defined subroutine (UserPtfmLd) in FAST as a 6×6 restoring matrix. These values are calculated using the method presented in (Wang and Sweetman, 2012b), but linearized near the average tilt angle and tuned to reproduce the correct natural frequencies. The initial conditions in all six DOFs of the tower are zero. The CM of the nacelle is not directly above the axis of the tower, so nacelle yaw motion changes the position of the CM of the system relative to the tower, which causes the tower motion. Figs. 5–6 show that the global motions of FAST and the MCM are virtually indistinguishable. The spin axis is initially parallel to the surge direction. The influence of the moving G_s can be seen clearly in both translational and rotational simulation results. For example, both pitch and surge are minimized (zero crossing) when the nacelle yaw angle is 90 deg (at 75 sec), while roll and sway are maximized. The observed yaw motion results from gyro moments associated with rotor spin coupled with roll and pitch.

8.2. Internal forcing from inverse dynamics

Figs. 7–8 show the internal forces and moments applied by the nacelle on the tower computed using inverse dynamics. This internal forcing is decomposed into the body-fixed (x_1, y_1, z_1) system. The nearly-constant internal force along the tower axis due to the topsides weight has been subtracted from this results. Results computed for the tower-top coordinate system in FAST have been transferred to the (x_1, y_1, z_1) system to enable direct comparison with the MCM; the comparison shows perfect agreement. The natural frequency of the tower in pitch motion dominates the time histories of internal forces shown in Fig. 5; these motions also affect the internal moments. The effect of nacelle yaw is also apparent in simulation results for both internal forces and moments.

9. Conclusions

A new multibody dynamics formulation method has been presented. The Momentum Cloud Method (MCM) can be used to obtain the explicit first-order decoupled EOMs convenient for numerical integration. The underlying concept is to directly apply the conservation of momentum to the entire multibody system. Various advantages of existing multibody dynamics formulations are combined in the MCM. Most notably, the calculation of internal forcing can be avoided and rederivation associated with adding a new body to the system is simplified. A set of 1-2-3 sequenced Euler angles are applied to describe the large-amplitude rotations of the floating base body, and all nonlinear effects can be preserved. More importantly, the MCM represents the conservation of momentum of entire system using only six basic EOMs, in which the translational and rotational inertial forcing are decoupled. The 6×6 mass matrix in the basic EOMs is actually composed of two 3×3 decoupled mass matrices: one for rotation and one for translation. This decoupling increases the efficiency of numerical integration dramatically. The selection of coordinates enables relatively simple expansion of the MCM to multibody systems with more complicated forms and connection joints. The unknown relative motion at joints can be solved by combining six basic EOMs with the constraint and control equations. The solutions to the constraint and control equations are used in the basic EOMs at each time step to represent kinematics of unknown relative coordinates. Inverse dynamics can also be applied to find internal forcing between any two contiguous bodies using the results of forward dynamics. The MCM is verified by critically comparing simulation results for a 6-body compliant wind turbine model with those computed using well-recognized multibody dynamics software.

10. Acknowledgements

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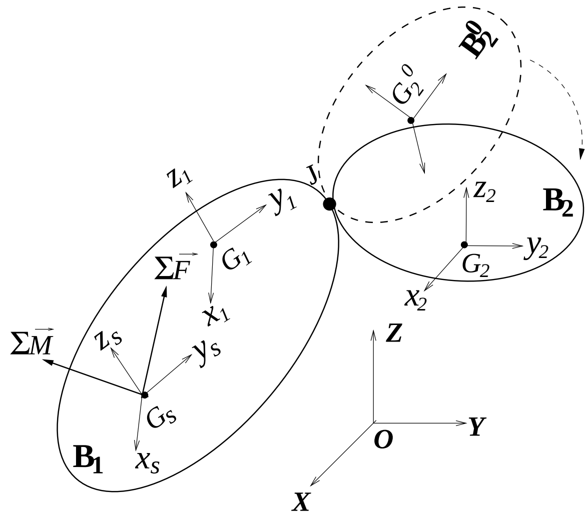

Figure 1: Two Bodies Connected by a Joint

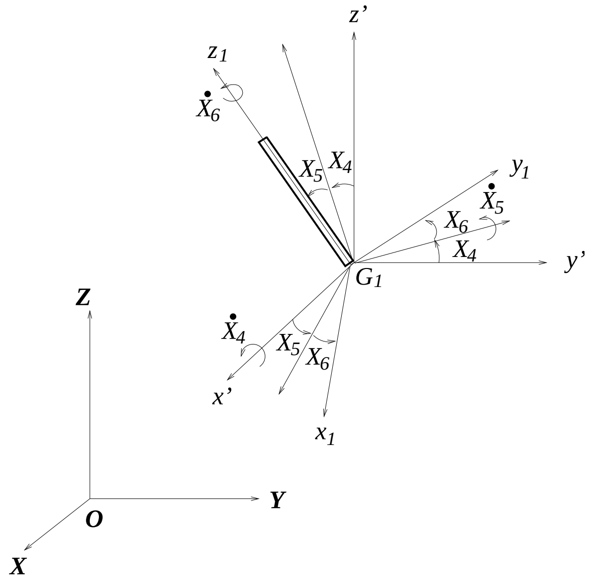

Figure 2: 1-2-3 sequenced Euler angles in terms of X_4, X_5 and X_6

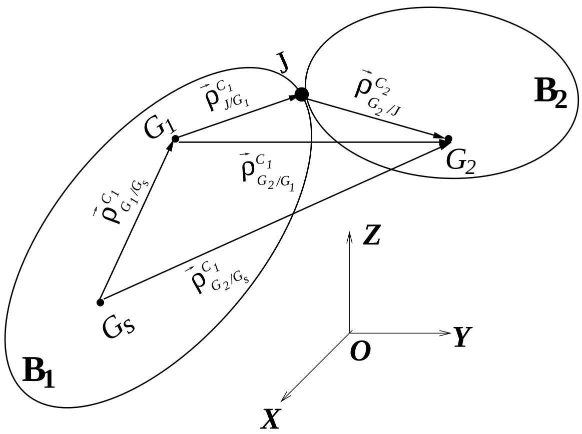

Figure 3: Calculation of Radius Vectors

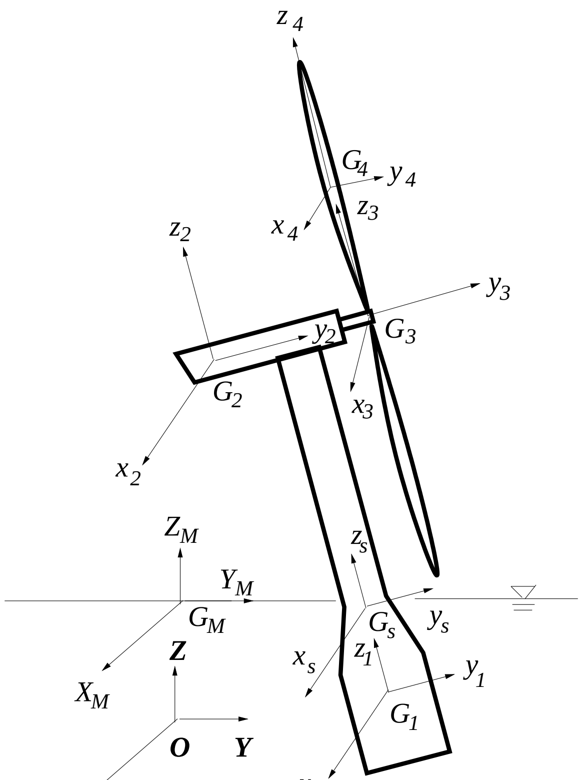

Figure 4: Coordinate systems in the application

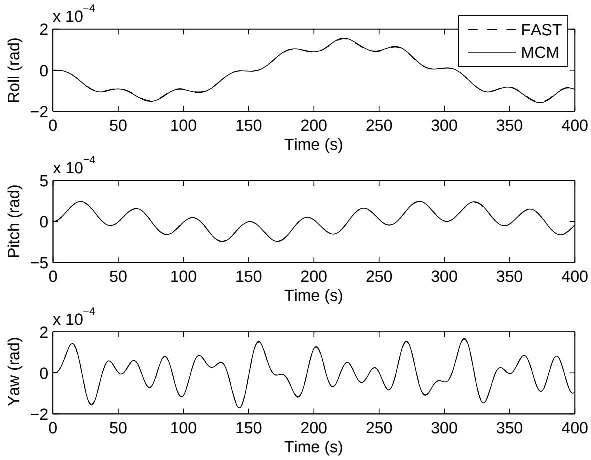

Figure 5: Rotation verification

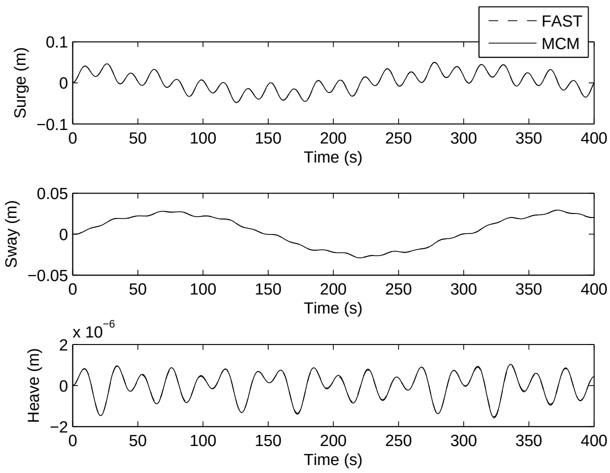

Figure 6: Translation verification

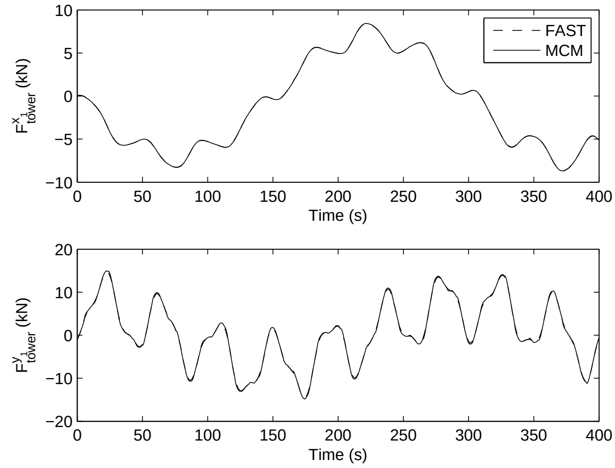

Figure 7: Internal forces applied by the nacelle on the tower

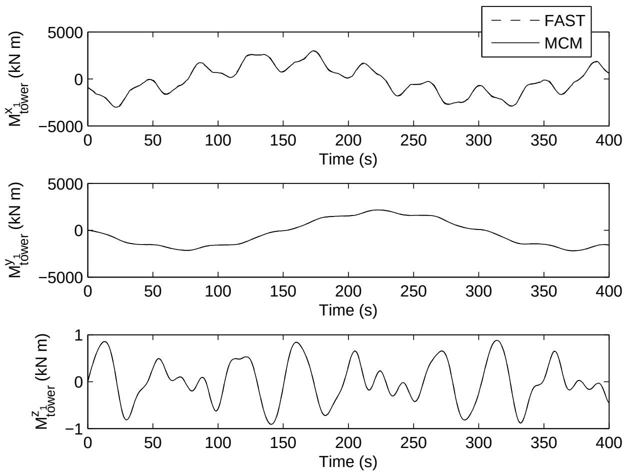

Figure 8: Internal moments applied by the nacelle on the tower