# $1^{st}$ TAMUG "Math Olympics"

## Problems & Solutions

from DJK

Each problem is stated, then a (usually short) solution is given, and sometimes an additional comment (or more) is made.

### Problem #1

Is there a tetrahedron every edge of which is opposite to some obtuse angle on a face of this tetrahedron? (By a tetrahedron we mean any polyhedron with 4 vertices, 6 edges, & 4 faces, so that each face is a triangle. And by an obtuse angle we mean one that is $> 90^{o}$.)

First, (following Dr. Qiu) note that no triangle can have more than 1 obtuse angle (as the sum of the angles in a triangle is $180^{o}$). Thus a tetrahedron (with 4 triangular faces) can have at most 4 obtuse angles, whence not every one of the 6 edges can be opposite to an obtuse angle.

Second, another approach (following Dr. Luxemburg) considers an edge of the tetrahedron that is of minimum length. Such an edge is included in two triangular faces, for which the other edges are at least as long, whence the angles opposite to this shortest edge can be argued not to be obtuse.

**Problem #2**

Show that there exists a nonzero integer $n$ such that the leading digits of the decimal expansion of $3^n$ are $10000$.

To see this, (following an idea of L. Luxemburg) take an increasing sequence of powers of $3$. Eventually there must be a pair, $a$ & $b$, of distinct exponents with $3^a$ & $3^b$ having the same $5$ lead digits, as there are only $9\times10^4$ such 5-digit sequences. That is, there is a 5-digit number $x_1x_2x_3x_4x_5 \equiv X$ such that $3^a = (X+\varepsilon_a)\times10^m$ & $3^b = (X+\varepsilon_b)\times10^n$ with $\varepsilon_a$ & $\varepsilon_b$ corrections which must be $<1$ (and $>0$). Now division of $3^b$ by $3^a$ leads to a number which is a power of $3$ and whose lead digits are necessarily close to 10000, though if $\varepsilon_a < \varepsilon_b$, the lead digits of this ratio can be 9999. In the first case we have achieved our goal, and in the second case, the inverse division (of $3^a$ by $3^b$) leads to a number with lead digits 10000. Thus in the first case $n = b-a$ is the requisite exponent, while in the second case it is $n = a-b$ (though only one of these gives a positive power).

If further $n > 0$ is demanded, we can continue the proof, addressing the second case, when $b > a$ but $3^{b-a}$ ends up with leading digits 9999. In this case there is some number, say $p$ $(\geq 4)$ of 9s, before there is another digit other than 9, whence we repeat the construction of the preceding paragraph seeking distinct powers $3^A$ & $3^B$ such that the first $p+1$ digits agree, and further such that the difference $B-A$ is $> b-a$. The condition that $B-A > b-a$ is achieved by simply considering a sequence of $10^{p+1}$ increasing powers such that successive exponents differ by at least $b-a$. Then for $B > A$, the power $3^{B-A}$ has its $p+1$

lead digits either a $1$ followed by $0$ s or else all $9$s. But in either case, the ratio $3^{B-A}/3^{b-a}$ gives our desired positive power of 3 with lead digits 10000.

The argument is evidently extendable to say: there is an exponent $n(q)$ such that $3^{n(q)}$ has leading digits in its decimal expansion being $1$ followed by $q$ zeroes (where $q$ is arbitrarily large).

The result may in fact be yet further extended to say: there is an $N$ such that $3^N$ has $M$ arbitrary leading digits $x_i$, $i=1$ to $M$, in its decimal expansion. To see this note from the preceding paragraph that there is an exponent $n(M+1)$ such that the decimal expansion of $3^{n(M+1)}$ has a lead digit $1$ followed by $M+1$ zeroes. Not all the remaining digits can be 0, since then $3^{n(M+1)}$ would be a power of $10$, with different prime factors. Thus $3^{n(M+!)}=(10^{M+1}+\Delta)\times 10^{B}$ with $0<\Delta<1$. Now raising $3^{n(M+1)}$ to successive powers, slowly ratchets up the first $M$ digits, indeed it may be verified that it ratchets them up $1$ by $1$.

More generally one might seek to establish something about uniformity of the distribution of the first $M$ digits of the base-$b$ expansion of $a^N$, as $N$ varies. One needs to take care when $a$ & $b$ are not relatively prime.

With the use of Maple, V. Rosenfeld finds $3^{700847}\cong 1.00001499184\times 10^{334389}$. To do this he sought an integer exponent $n$ with $3^n$ close to $10^N$, which is to say $3^n\cong 10^N$. Since the (non-integer) value $\{\log 10/\log 3\}n$ for $N$ would achieve exact equality, the quest is really just for rational approximations to $\log 10/\log 3$, such as there are ways of doing to sufficiently high accuracy.

## Problem # 3

Prove that if $b$ is an algebraic number, then so is $b/2$ . Here by an algebraic number $b$ we mean that there exists a nonzero polynomial with integer coefficients such tha $b$ is a root of this polynomial . That is, $b$ is an algebraic number if there are some integers $a_0, a_1, ..., a_n$ ($n \geq 1$) not all of whom are zero such that $a_n b^n + a_{n-1} b^{n-1} + ... + a_1 b + a_0 = 0$ .

To prove this, suppose that $b$ satisfies this condition of being algebraic, and rewrite its associated polynomial relation as

$$a_n \cdot 2^n \left(b / 2\right)^n + a_{n-1} \cdot 2^{n-1} \left(b / 2\right)^{n-1} + ... + a_1 \cdot 2(b / 2) + a_0 = 0 \text{,}$$

But here $a_n 2^n, a_{n-1} 2^{n-1}, ..., a_0$ are all integers, say $a_i' = 2^i a_i$ , $i = 0,1,2,...,n$ , and not all of the $a_i'$ are zero. That is, $b/2$ satisfies

$$a_n'(b/2)^n + a_{n-1}'(b/2)^{n-1} + ... + a_1'(b/2) + a_0' = 0$$

and so, $b/2$ is an algebraic number.

To frame a more general question, note that $1/2$ (by which we multiplied $b$) is itself an algebraic number. Thus one might more generally wonder whether $b \cdot c$ is an algebraic number when both $b$ & $c$ are.

Such algebraic numbers are an extension of rational numbers, since every rational number $p/q$ is obviously a root of the simple polynomial $p - qx$ . It is a fundamental area of study to determine the characteristics of this extended class of algebraic numbers.

## Problem #4

Given a collection of $n$ people such that everybody in this collection is a friend of exactly 25 others. Can $n$ be 103? Can $n$ be 104?

We assume that friendship is a 2-way relation, so that if A is a friend of B, then B is a friend of A. Think of each friendship relation as two 1-way relations, from A to B and from B to A. Thence for $n$ people each a friend to 25 others, there would need to be a number $25 \times n$ of 1-way relations, which would need to be combined into 2-way friendship relations, and which only might be done if $n$ is an even number. Thus $n$ cannot be 103. To conclude that $n$ can be 104, we display an example of such friendship pairing: Imagine the 104 people arranged cyclically, with each person being a friend, first to each of the 12 preceding people, second to the 12 succeeding people, and third to the opposite person on the cycle (there being an "opposite" when $n$ is even).

A more general question is to consider $n$ people each with $m$ friends. If $m$ is odd, then the considerations of the preceding paragraph indicate that necessary and sufficient conditions for this to be possible are that $n$ be even and $n > m$. If $m$ is even, say $m = 2p$, then one may imagine that each person in a cycle is a friend to the $p$ preceding and the $p$ succeeding people in the cycle, so that the only condition on $n$ is that $n > m$.

**Problem #5**

Let $p?$ denote the product of all prime numbers $\leq p$. Then if $p$ is a prime number $\geq 2$, prove that there is a prime number $q$ such that $p < q < p?$. (Here by a prime number we mean an integer $>1$ which has no positive integer divisors other than itself & $1$.)

A solution suggested by L. Luxemburg, starts considering the number $p?-1$ and viewing it as a candidate to be a $q$ in this range. Now no prime $p_i \leq p$ can divide $p?-1$ as all these primes divide $p?$ (which is just $1$ more than $p?-1$). Thus if $p?-1$ is not itself prime it must be divisible by a prime $p'$ which is not $\leq p$. That is, either $p?-1$ or a $p' > p$ is a prime and in addition this prime is found strictly between $p$ & $p?$.

One might note that the construction here seems related to Euclid's proof of the infinitude of primes. [This proceeds by imagining what it would mean if there were a finite number, say $N$, of primes. Then letting these presumed primes be $p_1, p_2, p_3, ..., p_N$, one forms their product and adds 1, to obtain a number $p_1 \times p_2 \times p_3 \times ... \times p_N + 1$, which cannot have these – or any other integers as divisors. That is, this new number satisfies the conditions of being a prime, contrary to our presumption that we had already listed the finite number of primes. Evidently our presumption that there is a finite number of primes is at fault, so that there must be a never-ending list of them – *i.e.*, we might say there is an "infinite" number of them.]

**Problem #6**

Is $2^{2009}-4$ divisible by $7$ ? Or equivalently, is $(2^{2009}-4)/7$ an integer?

First, (following Dr. Qiu) note that

$$2^{2009}-4=2^{3\cdot 669+2}-4=8^{669}\cdot 4-4=4\{(7+1)^{669}-1\}$$

Then use the binomial expansion to give $(7+1)^{669}$ as a sum of terms involving $7^m\times 1^{669-m}$ with $m=0,1,2,3,...,669$. That is,

$$(7+1)^{669}=C_0\times 7^0+C_1\times 7^1+C_2\times 7^2+...+C_{668}\times 7^{668}+C_{669}\times 7^{669}$$

with each coefficient $C_m$ being an integer (as it is the number of ways that one can choose $m$ factors of $7$ & $669-m$ factors of $1$ contributing to the product $7^m\times 1^{669-m}$). Here all of the terms other than the $m=0$ term are clearly divisible by $7$, but in the expression $(7+1)^{669}-1$, this $m=0$ term (with $C_0=1$) is exactly cancelled by the $-1$. Thus $(7+1)^{669}-1$ is a sum of powers $7^m$ with integer coefficients, for $m=1$ to $669$, and therefore is divisible by $7$, as also is $4\{(7+1)^{669}-1\}=2^{2009}-4$.

Seemingly other problems such as whether $2^{2009}-4$ is divisible by another particular integer $n$ might be more challenging.

**Problem #7**

Three brothers, Steve, John and Bill each got an inheritance in horses. Steve got one half of all the horses, John got 1/3 and Bill got $1/7$. After they received their horses, there was one horse left which went to the executor of the will. How many horses did each brother get?

To solve this note that $1-\frac{1}{2}-\frac{1}{3}-\frac{1}{7}=\frac{1}{42}$ of the inheritance going to the executor is 1 horse. So there are 42 horses in total. Steve got $\frac{1}{2}\cdot 42=21$ horses; John got $\frac{1}{3}\cdot 42=14$ horses; and Bill got $\frac{1}{7}\cdot 42=6$ horses.

This is somewhat reminiscent of a story where the father leaves a will giving 1/2 of his horses to the first son, 1/3 to the second son, and 1/9 to the third son. But in this (different) case there are just 35 horses, and the question is how to resolve the situation as none of the sons want the horses to be sliced up. To avoid slicing, the executor of the will offers to donate an additional horse of his own to the herd, to bring the herd up to a total of 36 horses. Then the first brother gets $\frac{1}{2}\times 36=18$, the second brother gets $\frac{1}{3}\times 36=12$, and the third gets $\frac{1}{9}\times 36=4$, whence no horses are sliced up, and there are 2 left over, which the executor again takes. Each of the sons is pleased not only for avoiding horse-slicing but also for receiving more than his share would have otherwise been – and yet also the executor is pleased in gaining an extra horse.

**Problem # 8**

A first pump working continuously at its nominal rate can fill a swimming pool with water in 3 days, and a second pump working at its nominal rate can fill the same pool in 4 days. Suppose the first pump is pumping the water into the pool with its nominal rate, and the second pump is pumping the water out at the same time with its own nominal rate. How long will it take for these two pumps working together in this (in & out) arrangement to fill the pool?

The first pump fills $1/3$ of the pool in a day, and the second pump empties $1/4$ of the pool in a day. Thus in one day, they work together to fill $\frac{1}{3}-\frac{1}{4}=\frac{1}{12}$ of the pool. Hence it takes 12 days to fill the whole pool.

Another problem (which some people evidently answered) supposes that both pumps are used simultaneously to fill the pool, and then the question is how long does this take. In this case, in 1 day $\frac{1}{3}+\frac{1}{4}=\frac{7}{12}$ of the pool is filled, so that it takes $1/\frac{7}{12}=12/7$ days to fill the pool.

More complicated problems with a greater number of pumps working at different rates can be solved in an analogous fashion.

Reading through such problems and understanding the method of solution is a first step toward generation of such solutions on your own. Such practice before a contest is usually good preparation. And overall such exercise should help you develop logical and rational reasoning processes.