Practice problems for the Math Olympiad

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<Problem #1>

Is there a tetrahedron such that its every edge is adjacent to some obtuse angle for one of the faces?

Answer: No.

Definitions: In geometry, a tetrahedron (Figure 1) is a polyhedron composed of four triangular faces, three of which meet at each vertex. Here, a face is a polygon bounded by a circuit of edges, and usually including the flat (plane) region inside the boundary. An edge of the tetrahedron is the line segments joining two vertices. An angle is the figure formed by two rays sharing a common vertex in the same face. And the obtuse angles are angles larger than a right angle and smaller than a straight angle (between 90° and 180°).

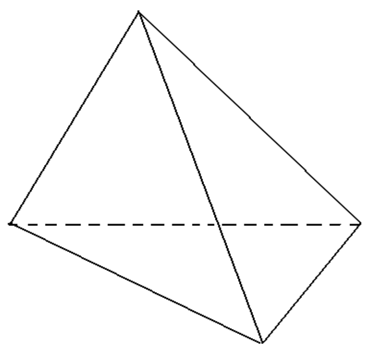

Figure 1

Proof: We will prove that there is no tetrahedron whose every edge is adjacent to some obtuse angle for one of the faces. Let us assume the contrary, i.e. that there is such a tetrahedron. Let E be its longest edge. i.e., the length of E is no shorter than that of any other edge. And let E be adjacent to an obtuse angle A in some triangular face F. We know that in any triangle the largest side is always opposite the largest angle, so the largest side S in F is located opposite the angle A. This side S is longer than E, so we came to a contradiction that E is not shorter than any other edge. This contradiction proves the theorem.

<Problem #2>

Solve the following system of equations (in real numbers):

$$x^3 + y^3 = 1$$

$$x^4 + y^4 = 1$$

Solution: Solving a system of equations of x and y means we need to find all the real pairs (x, y)'s satisfying both the following equations (2.1) and (2.2).

$x^3 + y^3 = 1$ (2.1)

$x^4 + y^4 = 1$ (2.2)

(a) At first, we observe that $-1 \le x \le 1$ and $-1 \le y \le 1$ from (2.2), since both x^4 and y^4 are nonnegative.

$x^4 \ge 0$,

$y^4 = 1 - x^4 \le 1$,

Then, $-1 \le y \le 1$.

And similarly, we get $-1 \le x \le 1$.

(b) It is obvious that $(0,1)$ and $(1,0)$ are two sets of solutions. Now we can consider the solutions in the remaining range when $-1 < x < 0$ and $0 < x < 1$.

(c) When $-1 < x < 0$, $x^3 < 0$, so $y^3 = 1 - x^3 > 1$ by (2.1), that means $y > 1$ which contradicts (a). So $0 < x < 1$.

(d) When $0 < x < 1$, $0 < x^3 < 1$, so $0 < y^3 < 1$ by (2.1), that means $0 < y < 1$.

Now we know $0 < x < 1$ and $0 < y < 1$,

So, $x^3 > x^4$ and $y^3 > y^4$

Hence, $x^3 + y^3 > x^4 + y^4$. Then it is impossible to have both $x^3 + y^3$ and $x^4 + y^4$ equal to 1.

So, $(0,1)$ and $(1,0)$ are the only solutions in real numbers.

<Problem #3>

Solve the following equation in integers:

$$x^3 = 2y^3 + 4z^3$$

Solution: The integers are formed by the natural numbers including 0 (0, 1, 2, 3, ...) together with the negatives of the non-zero natural numbers (−1, −2, −3, ...). Solving an equation of x, y and z in integer means we need to find all the integer triples $(x, y, z)'s$ satisfying the following equation (3.1).

$$x^3 = 2y^3 + 4z^3 \qquad (3.1)$$

It is easy to find $(0,0,0)$ is a solution. And we will prove there is no other solution in integers.

Assume that $|x| + |y| + |z|$ is the smallest positive integer for which an equation (3.1) is true. It is obvious that x is even, therefore $x = 2t$ for some integer t. This implies that

$$8t^3 = 2y^3 + 4z^3 \qquad (3.4)$$

Dividing (3.4) by 2, we get

$$4t^3 = y^3 + 2z^3$$

$$y^3 = 4t^3 - 2z^3$$

$$y^3 = 2(-z)^3 + 4t^3\,, \qquad (3.5)$$

which is the same type as (3.1) . Hence, $(y, -z, t)$ is also a solution of the original equation (3.1). And it is clear that

$$|y| + |-z| + |t| < |x| + |y| + |z|, \text{ if } x \neq 0.$$

This leads to a contradiction with the assumption of the minimality of $|x| + |y| + |z|$ unless $x = 0$.

Therefore, $x = 0$. And it follows that $0 = 2y^3 + 4z^3$ which implies that

$$0 = y^3 + 2z^3. \qquad (3.6)$$

From (3.6) it follows that y is even and $y = 2k$ for some integer k. Substituting this into (3.6) we get $0 = 8k^3 + 2z^3$ and $0 = 4k^3 + z^3$.

The latter equation can be written as $z^3 = 2 \cdot 0^3 + 4(-k)^3$ which is of the same type as (3.1) and

$$|z| + |-k| + |0| < |x| + |y| + |z| \text{ , unless } x = 0 \text{ and } y = 0.$$

This again contradicts the minimality assumption unless $x = 0$ and $y = 0$. The latter two equalities and (3.1) imply $z = 0$. Therefore, the only solution is $x = 0$, $y = 0$, $z = 0$.

<Problem #4>

Solve the following equation in integers:

$$3^m - 2^n = 1$$

Definition: Modular Arithmetic means recycling of integers when they reach a fixed value, e.g., a 12 hour clock. or integers a, b, n, we write a=b(mod n), read “a is congruent to b modulo n”, if a-b is a multiple of n. e.g., 38=14(mod 12) because 38 -14 = 24 =2*12.

Solution: For this question, we can solve it by finding all solutions and proving there are no others.

$$3^m - 2^n = 1(4.1)$$

(a) First, we observe that m and n are positive integers, since

$2^n > 0$,

$3^m = 2^n + 1 > 1$,

So, $m > 0$.

So m is a positive integer, 3^m is a positive integer. So

$2^n = 3^m - 1$ is an integer, then n has to be a nonnegative integer. (a negative power of 2 is a proper fraction) . Moreover $n \neq 0$. (If $n = 0$, $3^m = 2$, which is impossible.)

So, both m and n are positive integers.

(b) Next, we notice that m=n=1 is a solution. Now let's assume $n > 1$.Then

$$2^n = 0 (\text{mod}\, 4)$$

From (4.1), we have

$$3^m = 2^n + 1 = 1 (\text{mod}\, 4)$$

So, $m = 2k$, where k is a positive integer.

Now, we have $2^n = 3^{2k} - 1$.

Factoring it, we get

$$2^n = (3^k + 1)(3^k - 1)$$

So the integers $(3^k - 1)$ and $(3^k + 1)$ are both positive powers of 2, and they are 2 apart. So the only possibilities are $3^k - 1 = 2$ and $3^k + 1 = 4$. Hence $k = 1$. This implies $m = 2, n = 3$.

The solutions are m=n=1 and $m = 2, n = 3$.

<Problem #5>

Prove that if a middle lane of a quadrangle is equal to half the sum of its sides, then the quadrangle is a trapezoid, i.e. given a quadrangle ABCD and the middle of AB is H, the middle of CD is K. Then if HK is ½ of BC+AD, then ABCD is a trapezoid, i.e. BC is parallel to AD

Definition: A trapezoid (Figure 2) is a quadrilateral with two sides parallel. The middle lane is the line segment joining the middle points of two nonparallel sides.

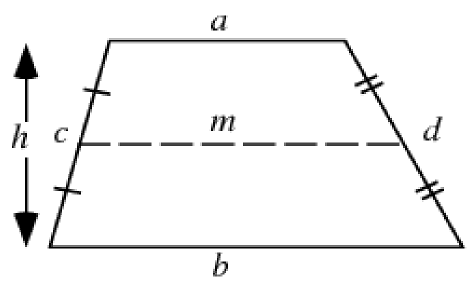

Figure 2

Proof: Assume ABCD is not a trapezoid, i.e. AD, HK and BC are not parallel. Then, we can draw AD' // HK and BC' // HK. Extend AD' such that D'S=BC'. And connect DS.

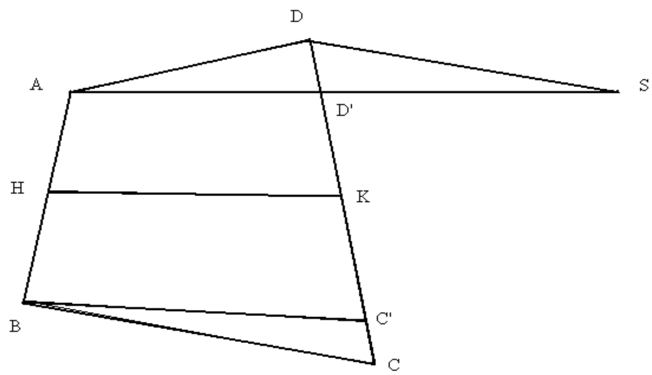

Figure 3

From AD’ // HK ,BC’ // HK and H is the midpoint of AB, we know ABC’D’ is a trapezoid, so

$$HK = \frac{1}{2}(AD' + BC') \qquad (5.1)$$

and D’K=C’K.

It is given that

$$HK = \frac{1}{2}(AD + BC) \qquad (5.2)$$

and DK=CK.

And DD’=DK-D’K, CC’=CK-C’K. So DD’=CC’.

From AD’ // HK // BC’, <DD’A=<KC’B. So <DD’S=<CC’B, because they are supplementary angles of <DD’A and <KC’B respectively.

Now we know DD’=CC’, D’S=C’B and <DD’S=<CC’B, so Triangle DD’S is congruent to Triangle CC’B. Hence BC=DS.

From (5.1) and (5.2), we get

AD’+BC’=AD+BC

By the congruence of BC’=SD’ and BC=DS, we have

AD’+SD’=AD+DS

i.e. AS=AD+DS.

This can happen only if A, D, S are on a line, that means AD // HK //BC. So ABCD is a trapezoid.

<Problem #6>

Given an increasing sequence of prime numbers $a_1, \dots, a_p$ forming an arithmetic progression, let p be a prime number and let $a_1 > p$. Prove that the difference $d = a_2 - a_1 = a_3 - a_2 = \cdots = a_p - a_{p-1}$ is divisible by p.

Definition: In Mathematics, a sequence is an ordered set of numbers. An increasing sequence is a sequence such that each element (the number in a sequence) is bigger than the element before it.

Definition: A prime number is a positive integer that is bigger than 1 and has no positive integer divisors other than 1 and itself. For example, 2,3,5,7, etc. An arithmetic sequence is a sequence such that the difference of any two successive elements is a constant.

Definition: An arithmetic progression is a sequence with common difference d:

$a, a + d, a + 2d, \dots$ etc. For example, 1,5,9,13,17,21,25 (here $d = 4, a = 1$)

Proof: For the arithmetic progression: $a_1, \dots, a_p$, there is a common difference d, such that

$$a_1 = a_1, a_2 = a_1 + d, a_3 = a_1 + 2d, \dots, a_p = a_1 + (p-1)d \qquad (6.1)$$

Let us prove that the numbers $a_1, \dots, a_p$ have different remainders after division by p. Assume the contrary, i.e. that there exist two of these numbers $a_i = a_1 + (i-1)d \; and \; a_j = a_1 + (j-1)d$, $i > j$ which have the same remainders. In this case, $a_i - a_j = (i-j)d$ is divisible by p. However, $i - j \geq 1$ and $i - j$ is smaller than p, therefore, $i - j$ is not divisible by p and by our assumption d is not divisible by p. Since p is prime, the product $(i-j)d$ is not divisible by p and we come to a contradiction. This proves that p numbers: $a_1, \dots, a_p$ have different remainders when divided by p and none of them is divisible by p because they are prime and are all greater than p. This means that all of them have nonzero remainders after division by p. But there are only $p - 1$ different remainders after the division by p which are not equal to zero. This is again a contradiction, because we have $p > p - 1$ such numbers, namely $a_1, \dots, a_p$. So d is divisible by p.

<Problem #7>

Prove that there are infinitely many positive integers which are not representable as a sum of cubes of two other positive integers.

Proof: Let us fix a positive integer n and let $1 \le x \le n^3$. Let us assume the positive number x can be represented as a sum of two cubes of two positive integers, i.e. $x = a^3 + b^3$. Then it is clear that

$$1 \le a \le n \text{ and } 1 \le b \le n$$

So there are n distinct a for $1 \le a \le n$ and there are n distinct b for $1 \le b \le n$. Therefore, there are no more than n^2 such pairs of a, b for which $x = a^3 + b^3 \le n^3$. And there are n^3 distinct x such that $1 \le x \le n^3$. So there are at least $n^3 - n^2$ integers not representable as a sum of cubes of two positive integers. Since $n^3 - n^2$ is a polynomial of positive degree with a leading coefficient 1, it can be made arbitrary large by making n sufficiently large. This proves the theorem.

<Problem #8>

Show that a positive integer is divisible by 9 if and only if the sum of its digits is divisible by nine.

Proof: The n-digit positive integer

$$\overline{a_n a_{n-1} ... a_2 a_1 a_0} = a_n \times 10^n + a_{n-1} \times 10^{n-1} + ... + a_2 \times 10^2 + a_1 \times 10 + a_0, \qquad (8.1)$$

where $a_0, a_1, a_2, ..., a_{n-1}, a_n$ are digits(meaning integers from 0 to 9) and $a_n \ne 0$.

Rewriting (8.1) by using $10^i = 1\underbrace{00...0}_{i} = \underbrace{99...9}_{i} + 1$, we get

$$\begin{aligned} &\overline{a_n a_{n-1} ... a_2 a_1 a_0} \\ &= a_n \times \left(\underbrace{99...9}_{n} + 1 \right) + a_{n-1} \times \left(\underbrace{99...9}_{n-1} + 1 \right) + ... + a_2 \times (99+1) + a_1 \times (9+1) + a_0 \\ &= a_n \times \underbrace{99...9}_{n} + a_{n-1} \times \underbrace{99...9}_{n-1} + ... + a_2 \times 99 + a_1 \times 9 + \left(a_n + a_{n-1} + ... + a_2 + a_1 + a_0 \right) \end{aligned}$$

Since $a_n \times \underbrace{99...9}_{n} + a_{n-1} \times \underbrace{99...9}_{n-1} + ... + a_2 \times 99 + a_1 \times 9$ is obviously divisible by 9,

$\overline{a_n a_{n-1} ... a_2 a_1 a_0}$ is divisible by 9 if and only if the sum of its digits $a_n + a_{n-1} + ... + a_2 + a_1 + a_0$ is divisible by 9.

<Problem #9>

Factor the following polynomials as a product of two other polynomials of smaller degrees:

(a) $x^{10} + x^5 + 1$

(b) $x^8 + x^4 + 1$

Definition: A polynomial is a mathematical expression involving a sum of powers in one or more variables multiplied by coefficients. A polynomial in one variable x with constant coefficients is given by $a_n x^n + ... + a_2 x^2 + a_1 x + a_0$.

Formula: $a^{2n+1} - b^{2n+1} = (a-b)(a^{2n} + a^{2n-1}b + \cdots + b^{2n})$ (9.1)

When $n = 1$, it becomes $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

When $n = 2$, it becomes $a^5 - b^5 = (a-b)(a^4 + a^3 b + a^2 b^2 + ab^3 + b^4)$

Solution: (a) Applying the formula (9.1) with $a = x^5, b = 1, n = 1$, we have

$$(x^5)^3 - 1^3 = (x^5 - 1)\left((x^5)^2 + x^5 \cdot 1 + 1^2\right)$$

$$x^{15} - 1 = (x^5 - 1)(x^{10} + x^5 + 1)$$

Dividing $x^5 - 1$ from both sides, we get

$$x^{10} + x^5 + 1 = \frac{x^{15} - 1}{x^5 - 1} \qquad (9.2)$$

And applying the formula (9.1) with $a = x^3, b = 1, n = 2$, we have

$$x^{15}-1=(x^3-1)(x^{12}+x^9+x^6+x^3+1)$$

So, $x^{15}-1$ is divisible by x^3-1. Now, we observe that x^5-1 isn't divisible by x^3-1.

Therefore by (9.2), $x^{10}+x^5+1$ has to be divisible by x^3-1.

Now we just perform long division to get $\frac{x^{10}+x^5+1}{x^3-1}=x^2+x+1$. So,

$$x^{10}+x^5+1=(x^3-1)(x^2+x+1).$$

(b) $x^8+x^4+1=\left(x^4+1\right)^2-x^4=(x^4+1-x^2)(x^4+1+x^2)$ by the difference of square.

<Problem #10>

Given three infinite sequences of positive integers,

$$a_1, a_2, a_3, \dots$$

$$b_1, b_2, b_3, \dots$$

$$c_1, c_2, c_3, \dots$$

Prove that there exist 3 indices p, q and r such that $p < q < r$ and

$$a_p \le a_q \le a_r$$

$$b_p \le b_q \le b_r$$

$$c_p \le c_q \le c_r$$

Definition: An infinite sequence of positive integers is a function from {1, 2, ...} to positive integer set in the notation $a_1, a_2, a_3, \dots$.. Here the elements in {1, 2, ... } are indices of the sequence. A sequence is bounded means there is a positive number N, such that all the elements a_n (n in {1, 2, ...}) in the sequence is bounded by N. i.e. $|a_n| \le N$, for all n in {1, 2, ...}.

Proof: (a) Let $X = \{x_n : n = 1,2,...\}$ be a sequence of positive integers and J be a set of indices then X(J) is a subsequence of X consisting of numbers in the sequence with these indices. First we prove that

$X = \{x_n : n = 1,2,...\}$ has an infinite subsequence which is non-decreasing. (10.1)

If X is bounded by a positive integer N, then, because there is a finite number of positive integers $\leq N$, there is a positive integer $x \leq N$ such that there are infinitely many numbers in X which equal to x. So a subsequence consisting of all numbers which are equal x is non-decreasing. That proves (10.1).

If X is unbounded, then we can choose a subsequence which is increasing by choosing a greater number at every subsequent step. That proves (10.1) for this case.

(b) Now let $A = \{a_n : n = 1,2,...\}$, $B = \{b_n : n = 1,2,...\}$, $C = \{c_n : n = 1,2,...\}$. We can use (10.1) to find an infinite set of indices J1 such that A(J1) is non-decreasing. Then we can use (10.1) again to find an infinite subsequence B(J2) in a sequence B(J1) which is non-decreasing and an infinite subsequence C(J3) in C(J2) which is also non-decreasing. It is clear that A(J3), B(J3) and C(J3) are all infinite subsequences and non-decreasing. So any 3 indices p, q, r in J3 will satisfy

$$a_p \leq a_q \leq a_r$$

$$b_p \leq b_q \leq b_r$$

$$c_p \leq c_q \leq c_r$$

<Problem #11>

Prove that $log_2 3$ and $\sqrt{2}$ are irrational.

Definition: A <u>rational number</u> is any number that can be expressed as the quotient or fraction a/b of two integers, with the denominator b not equal to zero. A real number that is not rational is called <u>irrational.</u>

Proof: (a) Let's assume $\log_2 3$ is rational, then there exist integers p and q, such that

$$\log_2 3 = \frac{p}{q} \qquad (11.1)$$

Since $\log_2 3$ is positive, we can assume both p and q to be positive integers.

Taking the power of 2 on both sides of (11.1), we get

$$2^{\log_2 3} = 2^{\frac{p}{q}}$$

$$3 = 2^{\frac{p}{q}}$$

Raising both sides by q-th power, we have

$$3^q = 2^p$$

However, 3^q is an integer not divisible by 2, so it can not be a positive power of 2 which makes the equality $3^q = 2^p$ impossible. This contradiction proves that the original assumption that $\log_2 3$ is rational is incorrect. So $\log_2 3$ is irrational.

(b) Assume $\sqrt{2}$ is rational, then there exist integers p and q, such that

$$\sqrt{2} = \frac{p}{q} \qquad (11.2)$$

We can also assume that p and q have no common divisors because then we can reduce the fraction $\frac{p}{q}$ to an irreducible one dividing both p and q by the largest common divisor.

Taking the square of both sides of (11.2), we get

$$2 = \frac{p^2}{q^2}$$

$$2q^2 = p^2 \qquad (11.3)$$

If p is odd then p^2 is also odd but p^2 is equal to $2q^2$ therefore p^2 is divisible by $\mathbf{2}$ and so p must be even, which means that $p = 2k$ for some integer k.

Substituting $p = 2k$ into (11.3), we get

$$2q^2 = p^2 = (2k)^2 = 4k^2$$

Dividing by 2, we get

$$q^2 = 2k^2,$$

which implies that q must also be even which leads us to a contradiction with the assumption that $\frac{p}{q}$ is irreducible. This means that the original assumption of rationality of $\sqrt{2}$ is also incorrect and $\sqrt{2}$ is irrational.