Taken from: www.math.ucla.edu/~radko/circles/lib/data/Handout-123-153.pdf

Pigeonhole Principle

The following general principle was formulated by the famous German mathematician Dirichlet (1805-1859):

Pigeonhole Principle: Suppose you have k pigeonholes and n pigeons to be placed in them. If $n > k$ (# pigeons > # pigeonholes) then at least one pigeonhole contains at least two pigeons.

In problem solving, the "pigeons" are often numbers or objects, and the "pigeonholes" are properties that the numbers/objects might possess.

1. In the movie "Cheaper by the Dozen," there are 12 children in the family.
   (a) Prove that at least two of the children were born on the same day of the week;
   Solution: There are 12 children (pigeons) which we are placing into 7 days of the week (pigeonholes), so by the PHP, some day of the week has two children.
   (b) Prove that at least two family members (including mother and father) are born in the same month;
   Solution: There are 14 family members (pigeons) and only 12 months they can be born in (holes), so some two family members must be born in the same month by the PHP.
   (c) Assuming there are 4 children's bedrooms in the house, show that there are at least 3 children sleeping in at least one of them.
   Solution: If no bedroom had at least 3 children, then each one would have 2 or fewer children, so with four bedrooms the number of children would be $\leq 4 \times 2 = 8$, but we know there are 12 children.

2. Pigeonhole Elementary School has 500 students. Show that at least two of them were born on the same day of the year.

   Solution: There are 500 students, and only 366 days they could have been born on, so by the PHP some two students were born on the same day.

3. There are 800,000 pine trees in a forest. Each pine tree has no more than 600,000 needles. Show that at least two trees have the same number of needles.

   Solution: In this problem, the 800,000 pine trees are the pigeons and the 600,000 different possible numbers of needles are the pigeonholes: we put a tree in hole # N if the tree has exactly N needles. Since there are more trees than possible numbers of needles, some two trees have the same number of needles, by the PHP.

Generalized Pigeonhole Principle: If n pigeons are sitting in k pigeonholes, where $n > k$, then there is at least one pigeonhole with at least n/k pigeons.

Example: If you have 5 pigeons sitting in 2 pigeonholes, then one of the pigeonholes must have at least $5/2 = 2.5$ pigeons—but since (hopefully) the boxes can't have half-pigeons, then one of them must in fact contain 3 pigeons.

1. Prove the Generalized Pigeonhole Principle.

   Solution: Assume there were *not* any pigeonhole with at least n/k pigeons. Then every hole has $< n/k$ pigeons, so the total number of pigeons is $< (n/k) \times (\# \text{ holes}) = (n/k) \times k = n$. But this says the number of pigeons is strictly less than n, and in fact there are exactly n pigeons, so our assumption that there were no pigeonhole with at least n/k pigeons must have been incorrect, and this means the Generalized Pigeonhole Principle is true.

2. There are 50 baskets of apples. Each basket contains no more than 24 apples. Show that there are at least 3 baskets containing the same number of apples.

   Solution: The baskets are the pigeons, and we place each of them in one of 24 pigeonholes according to how many apples are in it. Thus the ratio n/k of pigeons to pigeonholes is $50/24 = 2\frac{1}{12}$. By Generalized PHP there are at least this many baskets with the same number of apples, so there must be at least 3.

3. Show that among any 4 numbers one can find 2 numbers so that their difference is divisible by 3. (Avoid considering the cases separately. Use Pigeonhole Principle!)

   Solution: There are 3 possible remainders when we divide a number by 3 (0, 1, or 2). Thus by PHP, since we have 4 numbers, some two of them must have the same remainder on division by 3—so we can write these two as

   $$n_1 = 3k_1 + r \quad \text{and} \quad n_2 = 3k_2 + r,$$

   where r is the remainder on division by 3. Then the difference is

   $$\begin{aligned} n_1 - n_2 &= (3k_1 + r) - (3k_2 + r) \\ &= 3k_1 + r - 3k_2 - r \\ &= 3k_1 - 3k_2 \\ &= 3(k_1 - k_2), \end{aligned}$$

   which is divisible by 3.

4. Show that among any $n+1$ numbers one can find 2 numbers so that their difference is divisible by n.

   Solution: This is a more general version of the previous problem, and the solution is very similar. Here, since there are only n possible remainders on division by n, and we have $n+1$ numbers, by the PHP some two of them have the same remainder on division by n. Thus we can write these two as

   $$n_1 = nk_1 + r \quad \text{and} \quad n_2 = nk_2 + r,$$

where r is the remainder on division by n. Then the difference is

$$\begin{aligned} n_1 - n_2 &= (nk_1 + r) - (nk_2 + r) \\ &= nk_1 + r - nk_2 - r \\ &= nk_1 - nk_2 \\ &= n(k_1 - k_2), \end{aligned}$$

which is divisible by n.

5. Show that for any natural number n there is a number composed of digits 5 and 0 only and divisible by n.

   Solution: We will use the previous problem. We want to find a number divisible by n; the previous problem tells us that given any set of $n+1$ numbers, some two of them have a difference that's divisible by n. So we should try to find a set of $n+1$ numbers with the property that for any two of them, the difference is a number composed of digits 5 and 0 only. One possibility is the sequence of numbers $5, 55, 555, 5555, 55555, \ldots$, since the difference of any two of these will be some number of 5's followed by some number of 0's. So we can take the first $n+1$ numbers whose only digits are 5, and there must be some pair whose difference is composed of only 5's and 0's, and divisible by n.

6. Given 12 different 2-digit numbers, show that one can choose two of them so that their difference is a two-digit number with identical first and second digit.

   Solution: Again we use Problem 4 above. This time, since we have 12 numbers, there must be two of them whose difference is divisible by 11. But this difference can't have more than two digits, and since it's divisible by 11, it can't have fewer than two digits, so it must have exactly two digits. And any two-digit number divisible by 11 has identical first and second digit.

7. There are five points inside an equilateral triangle of side length 2. Show that at least two of the points are within 1 unit distance from each other.

   Solution: Draw a smaller equilateral triangle inside the first one by connecting the midpoints of its sides. This divides the triangle into four equilateral triangles of side length 2. Since there are 5 points, by the PHP some two points must lie in the same smaller triangle. But then these two points are in an equilateral triangle of side length 1, so they are within 1 unit distance of each other.

8. There are 10 (possibly overlapping) small line segments marked on a bigger line segment of length 1. If we add up the lengths of the marked segments, we get 1.1. Show that at least two of the marked segments have a common point. (Hint: Don't use the Pigeonhole Principle directly; instead use a similar argument to its proof.)

   Solution: Assume to the contrary the segments do not overlap. Then the total length covered by them is just the sum of their individual lengths, which is 1.1. But this is impossible, since they are contained in a segment of length 1. So our assumption to the contrary must be false, meaning the segments DO in fact overlap.

9. There are 13 squares of side 1 positioned inside a circle of radius 2. Show that at least 2 of the squares have a common point.

   Solution: If no 2 of the squares have a common point, then the total area they cover is equal to the sum of their individual areas, which is 13. On the other hand, a circle of radius 2 has are $\pi r^2 = \pi 2^2 = 4\pi \approx 12.56 < 13$, so it's impossible for the squares to cover an area of 13. Thus in fact some 2 of them must share a common point.