The First Texas A&M at Galveston Mathematics Olympiad, October 11, 2009.

1. Is there a tetrahedron whose every edge is a side opposite some obtuse flat angle on some face of this tetrahedron? (By a tetrahedron we mean any polyhedron with 4 vertices, 6 edges and 4 faces so that each face of it is a triangle).

2. Prove that there exists a nonzero integer n (positive or negative) such that the first nonzero digit of 3^n is 1 followed by 2 zeros.

3. Prove that if b is an algebraic number then so is $b/2$. (By an algebraic number b we mean a number which is a root of a nonzero polynomial with integer coefficients, i.e. the following is true:

$$a_n b^n + a_{n-1} b^{n-1} + \cdots + a_0 = 0 \; for\ some\ integers\ a_n , a_{n-1}, \ldots, a_0 \; (not\ all\ of\ them\ being\ zero\ )$$

4. Given a group of n people such that everybody in that group is a friend of exactly 25 others. Can n be 103? 104?

5. Let p? be a product of all prime numbers less than or equal to p. Let p be a prime number ≥ 5, prove that there is a prime number q such that $p < q < p?$.

Prime numbers are 2,3,5,7,11,… , i.e. integers >1 divisible only by 1 and itself.

6. Prove that $2^{2009} - 4$ is divisible by 7, i.e. that $(2^{2009} - 4)/7$ is an integer.

7. Three brothers, Steve, John and Bill each got an inheritance in horses. Steve got one half of all the horses, John got 1/3 and Bill got one seventh. After they received their horses, there was one horse left which went to the probate lawyer as a fee. How many horses did each brother get?

8. One pump working continuously at its nominal rate can fill the pool with water in 3 days, the second pump working at its nominal rate can fill the same pool in 4 days. Suppose the first pump is pumping the water into the pool with its nominal rate and the second pump is pumping the water out at the same time with its own nominal rate. How long will it take for these two pumps working together in this arrangement to fill the pool?

Helpful Note. The Dirichlet Principle

Sometimes the following consideration may help in solving some of the problems in math Olympics.

Suppose we have at least n+1 people living in n houses. The Dirichlet principle says that in this case, there should be a house with at least 2 people in it. Similarly, if we have 10 nonzero integers, then there are at least two of them which start with the same digit. If we have 91 integers >9, then at least two of them start with the same two digits (because there are only 90 possible two digits combinations such that the first digit is nonzero, 10,11,12,…,99).