The 7^{th} Annual TAMUG Mathematics Olympiad

1. Prove that there is no positive integer solutions to the equation $x^2 - y^2 = 2p$ where p is prime

   By prime number we mean an integer greater than one which is divisible only by one and by itself, the sequence of primes is infinite and starts with 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, ... etc.

2. Let O and P be two different points on a plane, and let L be an arbitrary straight line passing through O on that plane and let Q be the point on L such that QP is perpendicular to L. Describe the set of all points Q for various straight lines L passing through O.

3. Consider a Fibbonacci sequence: 1, 1, 2, 3, 5, 8, 13, 21,...

   Prove that there is a number divisible by 2015 in that sequence.

4. Solve the following equation in real numbers

   $$x^4 - 4x^3 + 6x^2 - 4x + y^4 + 2x^2y^2 - 4xy^2 + 2y^2 + 1 = 0$$

5. Consider Fibbonacci sequence: 1, 1, 2, 3, 5, 8, 13, 21,... What is the sum of the first hundred numbers of this sequence if $F_{101} = 354224848179261915075$, $F_{102} = 573147844013817084101$, and $F_{103} = 927372692193078999176$? *Here F_i is the i-th Fibbonacci number in that sequence. In other words, $F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5,$ etc.* You need to offer a proof, just an answer does not count. Do not use calculators.

6. $x = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \ldots}}}}}}}$

   The square root is repeated infinitely many times. Find x.

7. Two people left their cities at dawn at the same time. The person left city A walking towards city B and the second person has left city B walking toward city A. They met at noon and continued walking. The first person arrived at city B at 4 p.m. and the second person arrived at city A at 9 p.m. Assuming each person maintained a constant speed and walked along a straight line, determine when was the dawn.

8. Evaluate the following expression $(\log_2 x)(\log_x y)(\log_y z)(\log_z 8)$ assuming x, y, z are positive numbers not equal to 1. Justify your answer.

9. What is greater $e^{e^{e^{e}}}$ or $1000000^{1000000}$? Present a proof.

10. Given an arbitrary triangle, show that it can be cut by straight lines into 4 smaller and similar triangles such that a parallelogram can be formed from putting together these pieces without overlap.

11. Prove that out of 50 arbitrary integers, one can choose a group of 17 such that the difference between any two in this group is divisible by 3.