The 7^{th} Annual TAMUG Mathematics Olympiad

1. Prove that there is no positive integer solutions to the equation $x^2 - y^2 = 2p$ where p is prime

   By prime number we mean an integer greater than one which is divisible only by one and by itself, the sequence of primes is infinite and starts with 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, ... etc.

   Solution: Assume that there is positive integer solutions to the equation. Then $x^2 - y^2$ factors into $(x - y)(x + y)$ where $(x - y)$ and $(x + y)$ are integer. Since 2 is prime and p is prime, then either $(x - y) = 2$ or $(x + y) = 2$.

   If $(x - y) = 2$ then $(x + y) = p$ and $x = 2 + y$. Thus $2 + 2y = p$ and so p is even. Thus $p = 2$ since p is prime. Thus $y = 0$ which is a contradiction of y being a positive integer.

   If $(x + y) = 2$, then $x = 1$ and $y = 1$ since x and y are positive integer. Then $x^2 - y^2 = 0$ which is a contradiction since $2p \neq 0$.

   Since in every case we get a contradiction, it follows that there are no positive integer solutions. □

2. Let O and P be two different points on a plane, and let L be an arbitrary straight line passing through O on that plane and let Q be the point on L such that QP is perpendicular to L. Describe the set of all points Q for various straight lines L passing through O.

   Solution: *Coming Soon* □

3. Consider a Fibonacci sequence: 1, 1, 2, 3, 5, 8, 13, 21,...

   Prove that there is a number divisible by 2015 in that sequence.

   Solution: *Coming Soon* □

4. Solve the following equation in real numbers

$$x^4 - 4x^3 + 6x^2 - 4x + y^4 + 2x^2y^2 - 4xy^2 + 2y^2 + 1 = 0$$

   Solution: *Coming Soon* □

5. Consider Fibonacci sequence: 1, 1, 2, 3, 5, 8, 13, 21,... What is the sum of the first hundred numbers of this sequence if $F_{101} = 354224848179261915075$, $F_{102} = 573147844013817084101$, and $F_{103} = 927372692193078999176$? *Here F_i is the i-th Fibonacci number in that sequence. In other words, $F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5$, etc.* You need to offer a proof, just an answer does not count. Do not use calculators.

   Solution: Consider that $F_1 = 1 = F_3 - 1$, $F_1 + F_2 = 2 = F_4 - 1$, $F_1 + F_2 + F_3 = 4 = F_5 - 1$, and $F_1 + F_2 + F_3 + F_4 = 7 = F_6 - 1$. So it appears that $F_1 + F_2 + F_3 + ... + F_100 = F_{102} - 1$. We will prove that this must be true.

We know $F_1 = F_3 - 1$. Now we add F_2 to both sides and get $F_1 + F_2 = F_2 + F_3 - 1$. By the definition of the Fibonacci sequence $F_2 + F_3 = F_4$, thus $F_1 + F_2 = F_4 - 1$. Similarly we add F_3 to both sides and get $F_1 + F_2 + F_3 = F_3 + F_4 - 1 = F_5 - 1$. Thus if we continue in this fashion we will indeed get $F_1 + F_2 + F_3 + ... + F_{100} = F_{102} - 1$. Thus $F_1 + F_2 + F_3 + ... + F_{100} = 573147844013817084100$. □

6. $x = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + ...}}}}}}}$

   The square root is repeated infinitely many times. Find x.

   Solution: Note that $x = \sqrt{2 + x}$, thus:

$$
\begin{aligned}
x^2 &= 2 + x \\
x^2 - x - 2 &= 0 \\
(x - 2)(x + 1) &= 0
\end{aligned}
$$

   Thus $x = 2$ or $x = -1$. Since it is a positive square root, $x = 2$ only. □

7. Two people left their cities at dawn at the same time. The person left city A walking towards city B and the second person has left city B walking toward city A. They met at noon and continued walking. The first person arrived at city B at 4 p.m. and the second person arrived at city A at 9 p.m. Assuming each person maintained a constant speed and walked along a straight line, determine when was the dawn.

   Solution: Let r_1 be the rate of person 1 who is leaving city A and let r_2 be the rate of person 2 who is leaving city B. At noon, when they meet, person 1 has traveled $d_1 = r_1 t$ and person 2 has traveled $d_2 = r_2 t$. Note that person 1 traveled d_2 in 4 hours, so $d_2 = 4r_1$. Also, person 2 traveled d_1 in 9 hours, so $d_1 = 9r_2$. Thus:

$$
\begin{aligned}
r_2 t &= 4r_1 \quad \text{and} \quad r_1 t = 9r_2 \\
\frac{r_1}{r_2} &= \frac{t}{4} \quad \text{and} \quad \frac{r_1}{r_2} = \frac{9}{t}
\end{aligned}
$$

$$
\begin{aligned}
\frac{t}{4} &= \frac{9}{t} \\
t^2 &= 36
\end{aligned}
$$

   So $t = 6$ since t cannot be negative. Thus, they started out 6 hours before noon. Thus dawn is at 6 in the morning. □

8. Evaluate the following expression $(\log_2 x)(\log_x y)(\log_y z)(\log_z 8)$ assuming x, y, z are positive numbers not equal to 1. Justify your answer.

   Solution: First note that:

$$\begin{aligned}
\log_a b \log_b c &= n \\
a^{\log_a b \log_b c} &= a^n \\
\left(a^{\log_a b}\right)^{\log_b c} &= a^n \\
b^{\log_b c} &= a^n \\
c &= a^n \\
\log_a c &= n
\end{aligned}$$

So $\log_a b \log_b c = \log_a c$ which means

$(\log_2 x)(\log_x y)(\log_y z)(\log_z 8) = (\log_2 y)(\log_y z)(\log_z 8) = (\log_2 z)(\log_z 8) = (\log_2 8) = 3$

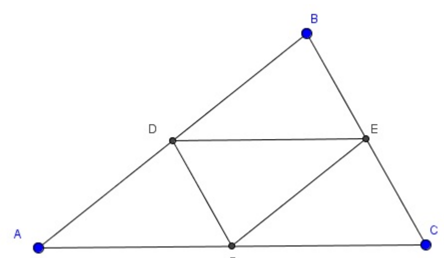
□

9. What is greater $e^{e^{e^{e^{e}}}}$ or $1000000^{1000000}$? Present a proof.

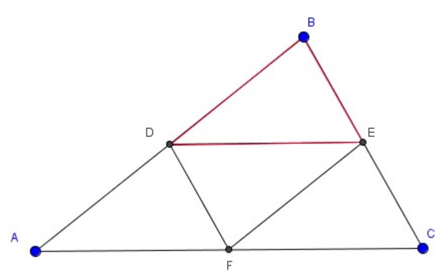
   Solution: *Coming Soon* □

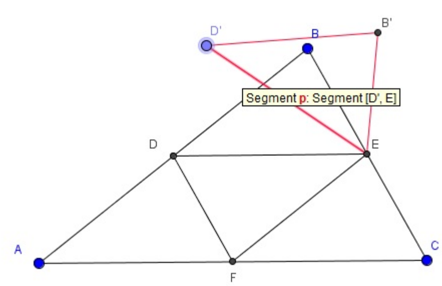
10. Given an arbitrary triangle, show that it can be cut by straight lines into 4 smaller and similar triangles such that a parallelogram can be formed from putting together these pieces without overlap.

    Solution: Consider an arbitrary $\triangle ABC$. Let D, E, and F be the midpoints of AB, BC, and AC respectively and connect them with segments. This creates the 4 smaller triangles.

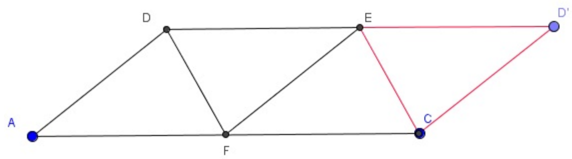


It is known that a segment connecting the midpoints of two sides of a triangle is parallel and half the length of the third side. Thus DE is parallel to AC and $DE = AF = CF$. Similarly $DF = CE = BE$ and $EF = AD = BD$. Thus the four triangular regions are congruent. Thus we will take $\triangle BDE$ and rotate it around E until $B'E$ aligns with CE.





Since DD' is parallel to AC, $DD' = AC$, and $AD = D'C$ the resulting figure is a parallelogram.



□

11. Prove that out of 50 arbitrary integers, one can choose a group of 17 such that the difference between any two in this group is divisible by 3.

    Solution: 50 arbitrary integer can be put into three groups:

    A integers divisible by 3

    B integers with remainder 1 when divided by 3

    C integers with remainder 2 when divided by 3

    By the pigeonhole principle, one of these groups has at least 17 members. If we try to distribute 50 integers evenly into three groups we get groups of size 16, 17, and 17. By the remainder theorem members of group A will have the form $a = 3n$, members of group B will have the form $b = 3n + 1$, and members of group C will have the form $c = 3n + 2$.

    If group A has 17 members, then for any two members we get $a_1 - a_2 = 3n_1 - 3n_2 = 3(n_1 - n_2)$ which is divisible by 3.

    If group B has 17 members, then for any two members we get $b_1 - b_2 = 3n_1 + 1 - (3n_2 + 1) = 3n_1 - 3n_2 = 3(n_1 - n_2)$ which is divisible by 3.

    If group C has 17 members, then for any two members we get $c_1 - c_2 = 3n_1 + 2 - (3n_2 + 2) = 3n_1 - 3n_2 = 3(n_1 - n_2)$ which is divisible by 3.

    Thus in every case a group with 17 member produces a difference of any two members that is divisible by 3. □