1. Between noon and the succeeding midnight, how many times do the hands of a clock form a angle? Give the reason.

   Solution: The minute hand moves at a rate of 360 degrees per hour (dpr) and the hour hand moves at a rate of 30 dpr. Thus the angle between the two hands is changing at a rate of 330 dpr (note that after the first hour the angle between them is 330 degrees in one direction and 30 degrees in the other).

   So we have the equation $330t = 180 + 360k$ where k is the number of times the minute hand crosses the hour hand (So in the first hour $k = 0$). Thus $t = \frac{6}{11} + \frac{12k}{11}$.

2. There is a solid rectangular box with dimensions $300 \times 200 \times 100$. Someone painted its entire surface red; then cut it into $1 \times 1 \times 1$ cubes. Find the numbers of cubes: (a) free of paint; (b) painted on exactly one side; (c) painted on exactly two sides; and (d) painted on exactly three sides.

   Solution: In a $300 \times 200 \times 100$ rectangular box there are a total of 6,000,000 $1 \times 1 \times 1$ cubes. Recall that a rectangular box has six sides, twelve edges, and eight corners.

   The cubes with exactly three painted sides must be on the corner. Thus there are 8 cubes with exactly three sides painted.

   The cubes with exactly two painted sides must be on the edges minus the corners. Of the twelve edges, four run the length of 300, 4 the length of 200, and four the length of 100. When we take off the corners that leaves 298 on a 300 edge, 198 on a 200 edge, and 98 on a 100 edge. Thus there are $4 * 298 + 4 * 198 + 4 * 98 = 2376$ cubes with exactly two sides painted.

   The cubes with exactly one side painted must be on the sides of the box minus the edge and corner cubes. Of the six sides, two are 300×200, two are 200×100, and two are 300×98. When we account for the edges and corners these sides become 298×198, two are 198×98, and two are 298×98 respectively. Thus there are $298*198*2+198*98*2+298*98*2 = 215224$ cubes with exactly one side painted.

   The cubes with no sides painted are completely contained in the box. When we remove the outer cubes, the ones with painted sides, we are left with a rectangular box that has dimensions $298 \times 198 \times 98$. Thus there are $298 * 198 * 98 = 5782392$ cubes with no side painted.

   Note that $8 + 2376 + 215224 + 5782392 = 6,000,000$ □

3. Find the digits such that when they are substituted for the letters below, you get a correct identity:

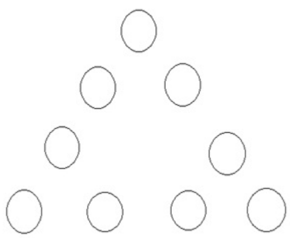
$$\begin{array}{ccccc} F & O & R & T & Y \\ + & & T & E & N \\ + & & T & E & N \\ \hline S & I & X & T & Y \end{array}$$

   Solution: *Coming Soon*

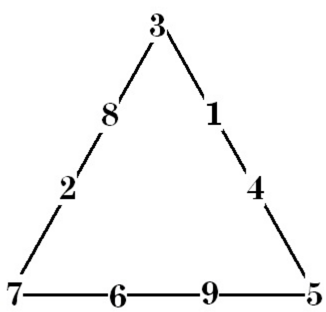
4. Show that if n and p are integers, then the product $n(n+1)...(n+p)$ is divisible by $p+1$.

Solution: We will assume from the product that $p > 0$. Note that for any integer k, every $|k|^{\text{th}}$ integer after k is divisible by k. In other words, $k \pm k$ is divisible by k, $k \pm 2k$ is divisible by k, $k \pm 3k$ is divisible by k, etc. Thus in any set of $|k|$ consecutive integers, one of the integers is divisible by k. There are $p+1$ consecutive integers in the product. Thus there is one integer that is divisible by $p+1$. Thus the product is divisible by $p+1$. $\square$

5. Put the numbers 1,2,3,4,5,6,7,8,9 into the following circles and make the sum of the numbers on each side of the triangle equal. You can use each number only once to fill all the circles.



Solution:



6. A train passes by an observer in t seconds and by a bridge of length L in s seconds. Find the speed of the train and its length in terms of t, s, and L. (Assume that the train passes through the bridge from the moment when the beginning of the locomotive enters the bridge until the moment when the end of the last car leaves the bridge)

Solution: *Coming Soon*

7. Show that no plane can intersect more than 4 edges of a tetrahedron.

Solution: *Coming Soon*

8. Find all triples of integers x, y, & z such that $x^3 + 3y^3 = 9z^3$. (Consider all possibilities for the integers, positive, negative, or 0.)

Solution: Note that $x = 0$, $y = 0$, & $z = 0$ is obviously a solution. Also, if two are 0, then clearly the third is also 0.

If $x = 0$, $y \neq 0$, & $z \neq 0$, then $y^3 = 3z^3$ and so y is a multiple of 3. Thus $y = 3a$ for some integer a and $27a^3 = 3z^3$. Therefore $9a^3 = z^3$ and so z is also a multiple of 3. Thus $z = 3b$ for some integer b and $9a^3 = 27b^3$. Consequently $a^3 = 3b^3$ and we see that we can continue this forever. Thus $x = 0$, $y \neq 0$ & $z \neq 0$ cannot happen. Similarly y or z cannot be 0 if there other two are not.

Suppose $x \neq 0$, $y \neq 0$, & $z \neq 0$, then x is a multiple of 3. Thus $x = 3a$ for some integer a and $27a^3 + 3y^3 = 9z^3$. Thus $9a^3 + y^3 = 3z^3$ and again we see that we can continue doing this forever. Thus $x \neq 0$, $y \neq 0$, & $z \neq 0$ cannot happen.

Thus the only solution is $x = 0$, $y = 0$, & $z = 0$.

9. Consider a sequence of numbers $a_1, a_2, ..., a_n$ each of which is either $+1$ or -1, and let us transform them into another sequence of $+1$s and -1s by multiplying a_1 by a_2, a_2 by a_3, etc. finally multiplying a_n by a_1.

   (a) Show that if $n = 2^k$ then after several transformations we should get all $+1$s.

   (b) Show that if n is odd then we can never get all $+1$s unless the original set consists of all $+1$s, or all -1s

   Solution: For any sequence $a = a_1, a_2, ..., a_n$ and any $m = 0, 1, 2, 3, ...$ let $K^m a$ denote the sequence obtained after m iterations of the transformation described in the problem. So