The Sixth Mathematics Olympiad at Texas A&M at Galveston

1. Given a cube with side length 1 and a sphere of radius 1.5 centered at one of the vertices of the cube. How many vertices of the cube lie inside the sphere? Explain your answer.

2. Solve an equation $5x^2 + 2y^2 - 6xy - 2x + 2y + 1 = 0$

3. Find the following sum: $\frac{1}{1+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{5}} + \frac{1}{\sqrt{5}+\sqrt{7}} + \frac{1}{\sqrt{7}+\sqrt{9}} + ... + \frac{1}{\sqrt{2013}+\sqrt{2015}}$

4. Two cubes have a common center and both have sides equal to 1. Prove that the common part of these two cubes have volume at least $\frac{\pi}{6}$.

5. Given a regular polygon with p sides, where p is a prime number. After rotating this polygon about its center by an integer number of degrees it coincides with itself. What is the maximal possible number for p?

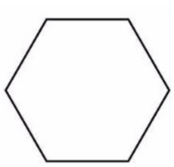
6. Find x given the following:

$$x = \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + ...}}}}$$

7. An ellipse is a figure in the plane, described by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where, in general, $a \neq b$. When $a = b$, the equation describes a circle with radius a. Therefore, a circle is also an ellipse. It is known that any two ellipses have no more than 4 points of intersection, i.e. no more than 4 points in common. Prove that a regular hexagon cannot be inscribed into an ellipse which is not a circle, i.e. into an ellipse given by the equation if $a \neq b$.



regular hexagon

ellipse with $a \neq b$

8. On TAMUG campus in the Fall of 2014 there are 2000 enrolled students and 50 courses offered. Prove that there are at least two females or two males which are born in the same month and are taking the same course.

9. Suppose it is given that on a University Campus of every three students at least two are acquainted with each other. Prove that if there are a total of 2001 students, then there is one student with whom at least a thousand other students are acquainted.

10. Given a polynomial $P(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$ whose all coefficients $a_n, a_{n-1}, ..., a_0$ are integers. If $P(x) = 5$ for 3 different integers a, b, and c, then there is no integer d for which $P(d) = 6$. (Hint: every polynomial $P(x)$ is a product $a_n(x - r_1)(x - r_2)...(x - r_n)$, where $r_1, r_2, ..., r_n$ are the roots of $P(x)$)

11. Find the smallest positive integer a such that $2014^{2014} - a$ is divisible by 7.