The Fourth Mathematics Olympiad at Texas A&M at Galveston
October 10, 2012

1. Given two intersecting circles of radii R_1 and R_2 such that the distance between their centers is d and a straight line tangent to both circles at points A and B. Find the length of the segment AB.

2. What are the last two digits of 2^{2012}?

3. Can a plane intersecting a cube result in a regular pentagon? Give the reasoning.

4. Given the Fibonacci sequence 1,1,2,3,5,8,13,21,...etc. show that there are more than 1,000 Fibonacci numbers smaller than 2^{1000}.

5. Let Q be a cube and let D be its diagonal connecting two vertices not belonging to the same face of the cube. What figure do we get if we cut the surface of the cube with the plane perpendicular to D and passing through the center of the cube?

6. Given 3 types of pumps: when a type-I pump pumps water into the empty pool it takes 3 hours to fill it, type-II pump takes 4 hours to fill it and type-III pump takes 5 hours. Also, the pumps pump the water out at the same speed as they pump it in. Suppose one has several pumps of each kind, some of the pumps of any type are set to pump water in and some of them are set to pump water out of the pool. Show that whatever the setup is, it is not possible to get them to fill the empty pool with water in exactly 7 hours if all pumps start and finish at the same time.

7. Suppose there are n boxes in which k balls are placed, where k and n are positive integers greater than 2. Given some positive integer r,
   (a) find the largest value of k which guarantees, that no matter how you place these k balls in n boxes, there would be at least one box containing no more than r balls.
   (b) find the largest value of k which guarantees, that no matter how you place these k balls in n boxes, there would be at least three boxes each containing no more than r balls.

8. A train shuttles between city A and city B at a uniform speed, spending negligible time at the stations and negligible time for acceleration and deceleration. A person living by the railroad tracks looks out the window at random times and waits until he sees the first train. He observes that on average, he sees the train coming from city B twice as often as from city A. What is the ratio of his distance from city B to his distance from city A?

9. Given a group of 101 telephones such that each telephone is connected with exactly two others in this group. Show that there is an odd number k such that $k > 2$ and a set of k telephones $T_1, T_2, ..., T_k$ from this group such that this set is connected in a circular fashion, i.e. first telephone T_1 is connected to T_2, T_2 is connected to T_3, etc., T_{k-1} is connected to T_k, and T_k is connected to T_1.

10. Given a rectangular tabular arrangement of numbers (a matrix) with n rows and m columns. Each number in this table is an integer, negative, positive, or zero. One is allowed to perform the following operations:

    (a) Simultaneously change the signs of all numbers in any particular column

    (b) Simultaneously change the signs of all numbers in any particular row

    Prove that after several such operations, we can get the sum of numbers in each row to be nonnegative and the sum of numbers in each column to be nonnegative at the same time.