# The 2nd Texas A&M at Galveston Mathematics Olympiad

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## Problems & Solutions

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Problem #1:

A runner passes 5 poles in 30 seconds. How long will he take to pass 10 poles? The consecutive poles are of equal distance from each other.

Solution:

Since it takes the runner 30 seconds to cover 4 spaces between the 5 poles, the rate of covering 1 space is 30/4 seconds. There are 9 spaces between 10 holes, so the time of passing 9 holes is $\frac{30}{4}\times 9 = 67.5$ seconds.

Problem #2:

A 12 hour clock shows 1 o'clock now. Exactly when will the hour and minute hands next coincide?

Solution:

Every minute the hour hand passes $\frac{1}{60}$ of the distance between 1 o'clock & 2 o'clock positions, which is $\frac{1}{12}$ of a full rotation, whereas the minute hand passes $\frac{1}{60}$ of the full rotation around the clock. After 1 o'clock, assume the hour hand and the minute hand next coincide $x$ minutes after 1 o'clock. Then the fraction of the distance around the clock (from the vertical position) is $\frac{1}{12}+\frac{1}{60}\cdot\frac{1}{12}x$ for the minute hand, while it is a fraction $\frac{x}{60}$ of a full rotation. Thus

$$\frac{1}{12}+\frac{1}{60}\cdot\frac{1}{12}x=\frac{1}{60}x,$$

which simplifies to

$$1+\frac{1}{60}x=\frac{1}{5}x$$

Solution for $x$ gives

$$x=\frac{60}{11}=5\frac{5}{11}.$$

So the hour hand and the minute hand will next coincide exactly $5\frac{5}{11}$ minutes after 1 o'clock ,or exactly at $1:05.\overline{45}$.

Problem #3:

There were originally 7 pieces of papers in a pile. Somebody picked up a piece and cut it into 7 pieces and put them back into the pile again. This was repeated many times. Then, somebody counted the pieces of the papers in the pile and got the number 2010. Prove that he made a mistake counting.

Solution:

There were originally 7 pieces of papers. Every time somebody cut and put back papers, he added 6 pieces of papers. So,

After his 1st cut and put back, he got $7+6=13$ pieces;

After his 2nd cut and put back, he got $13+6=7+2\times 6=19$ pieces;

……

After his 333rd cut and put back, he got $7+333\times 6=2005$ pieces;

After his 334th cut and put back, he got $7+334\times 6=2011$pieces.

So it is impossible to get 2010 pieces.

A simpler explanation: Since 2010-7=2003 is not divisible by 6, so it is wrong.

An even simpler explanation: the numbers of papers should always be odd.

Problem #4:

(a) What is the last digit of $2^{2010}$ ?

(b) What is the first digit of $2^{2010}$ ?

Solution:

(a) Let's observe the powers of 2:

$2^1=2$

$2^2=4$

$2^3=8$

$2^4=16$

$2^5=32$

...

The last digits of powers of 2 are 2, 4, 8, 6, 2, … with the cycle of 4. And 2010 has the remainder 2 when it is divided by 4. So the last digit of $2^{2010}$ is 4.

(b) Write $2^{2010}$ in the scientific notation, we have

$2^{2010}=c\times 10^n$ , where c is a decimal between 1 and 10,

and n is a positive integer.

Let $a$ be the first digit of integer $2^{2010}$ ($a$ can be 1,2,…,9), so $a\le c<a+1$. Then

$$a \times 10^n \le 2^{2010} < (a+1) \times 10^n$$

Take the common logarithm,

$$\log(a \times 10^n) \le \log(2^{2010}) < \log\left((a+1) \times 10^n\right)$$

$$n + \log a \le 2010 \cdot \log 2 < n + \log(a+1)$$

Estimate $2010 \cdot \log 2 = 605.07029...$

So, $n = 605$, from the whole part of the number above.

And, $a = 1$, since the decimal part of the number above is between $\log 1 = 0$ and $\log 2 = 0.301...$

So, the first digit of $2^{2010}$ is 1.

Problem #5:

Prove that $\frac{n(n+1)(2n+1)}{6}$ is an integer, for any integer n.

Solution:

Proving $\frac{n(n+1)(2n+1)}{6}$ is an integer is equivalent as proving' $n(n+1)(2n+1)$ is divisible by 6. And because $6 = 2 \times 3$, we will just need to prove $n(n+1)(2n+1)$ is divisible by both 2 and 3.

(a) Proof of divisibility of 2: if $n$ is even ($n$ is divisible by 2), then the product $n(n+1)(2n+1)$ is of course divisible by 2, we are done. If $n$ is odd, then $n+1$ is divisible by 2. So the product $n(n+1)(2n+1)$ is divisible by 2.

(b) Proof of divisibility of 3: if $n$ is divisible by 3, then we are done.

If $n$ has the remainder 1 when divided by 3, then by The Remainder Theorem, we have

$n = 3k + 1$, where k is an integer.

Then,

$$2n+1=2(3k+1)+1=6k+3=6(k+1)$$

is divisible by 3. So that $n(n+1)(2n+1)$ is divisible by 3.

If $n$ has the remainder 2 when divided by 3, then by The Remainder Theorem, we have

$n=3k+2$, where k is an integer.

Then,

$$n+1=3k+2+1=3k+3=3(k+1)$$

is divisible by 3. So $n(n+1)(2n+1)$ is divisible by 3.

Problem #6:

Let $f(x)=\frac{x}{\sqrt{1+x^2}}$, $f_n(x)=f(f(\ldots f(x)))$ is the nth composition of $f(x)$. Find $f_{99}(1)$.

(Composition or superposition of two functions $f(g(x))$ means: plug $g(x)$ as an argument into $f(x)$.)

Solution:

Let's observe the first several compositions:

$$f_1(x)=f(x)=\frac{x}{\sqrt{1+x^2}}$$

$$f_2(x) = f(f(x)) = \frac{\dfrac{x}{\sqrt{1+x^2}}}{\sqrt{1+\left(\dfrac{x}{\sqrt{1+x^2}}\right)^2}} = \frac{\dfrac{x}{\sqrt{1+x^2}}}{\sqrt{\dfrac{1+2x^2}{1+x^2}}} = \frac{x}{\sqrt{1+2x^2}}$$

$$f_3(x) = f(f_2(x)) = \frac{\dfrac{x}{\sqrt{1+2x^2}}}{\sqrt{1+\left(\dfrac{x}{\sqrt{1+2x^2}}\right)^2}} = \frac{\dfrac{x}{\sqrt{1+2x^2}}}{\sqrt{\dfrac{1+3x^2}{1+2x^2}}} = \frac{x}{\sqrt{1+3x^2}}$$

We can use mathematical induction to show that

$$f_n(x) = \frac{x}{\sqrt{1+nx^2}}$$

Then,

$$f_{99}(1) = \frac{1}{\sqrt{1+99\times(1)^2}} = \frac{1}{\sqrt{100}} = \frac{1}{10}.$$

Problem #7:

Numbers *p* and $8p^2+1$ are both primes. Prove $8p^2-p+2$ is prime.

(Prime number is an integer $>1$ which is divisible only by 1 and itself.)

Solution:

When $p=3$, $8p^2+1=8(3)^2+1=73$ is prime. So $8p^2-p+2=8(3)^2-3+1=71$ is prime.

Next, we will prove there are no greater choices of prime $p$ satisfying the conditions.

$p$ is prime and $p \neq 3$, then $p$ isn't divisible by 3. That means $p$ must have the remainder 1 or 2 when divided by 3.

First, if $p$ had the remainder 1 when divided by 3, then by the Remainder Theorem,

$p = 3k + 1$, where $k$ is an integer

Then,

$$8p^2 + 1 = 8(3k+1)^2 + 1 = 8(9k^2 + 6k + 1) + 1 = 72k^2 + 48k + 9 = 3(24k^2 + 16k + 3)$$

is divisible by 3, and hence it's not prime.

Second, if $p$ had the remainder 2 when divided by 3, then by the Remainder Theorem,

$p = 3k + 2$, where $k$ is an integer

Then,

$$8p^2 + 1 = 8(3k+2)^2 + 1 = 8(9k^2 + 12k + 4) + 1 = 72k^2 + 96k + 33 = 3(24k^2 + 32k + 11)$$

is divisible by 3, and hence it's not prime.

Problem #8:

Write down the first 3 significant digits of $\sqrt{0.999...9}$ with a hundred 9's. Show your work.

(Significant digits: the digits which start from the first non-zero digit after the decimal point.)

Solution: (Given by Dr. Luxemburg)

We will use the following two facts about inequalities:

(8.1) if $a > b$ and $b > c$ then $a > c$

(8.2) if $a > b$ and $c > 0$ then $ac > bc$

First of all, we need to show that:

(8.3) If a number $x$ is such that $0 < x < 1$ then

(8.4) $x < \sqrt{x} < 1$

Let us first prove that

(8.5) $\sqrt{x} < 1$

Indeed, assume the contrary i.e. that

(8.6) $\sqrt{x} > 1$

Then, multiplying both sides of (8.6) by $\sqrt{x}$ and using property (8.2) with $c = \sqrt{x}$ , we get

(8.7) $\sqrt{x}\sqrt{x} > \sqrt{x}$ or $x > \sqrt{x}$.

However, due to (8.6) $\sqrt{x} > 1$ , therefore, since $x > \sqrt{x}$ the rule (8.1) implies that

(8.8) $x > 1$

which contradicts the assumption that $x < 1$ . This contradiction proves (8.5). Multiplying both parts of inequality (8.5) by $\sqrt{x}$, and using the rule (8.2) we get

(8.9) $\sqrt{x} > x$

Therefore (8.4) follows from (8.8) and (8.9). From (8.5) it follows that $\sqrt{x}$ in decimal representation starts with 0. followed by some digits.

Let now $x = 0.9999\ldots9$ (with a hundred 9s). Then $x = 1 - 10^{-100}$.

If in the first 100 digits of $\sqrt{x}$ there is at least one which is less than 9 then $\sqrt{x} < x$ which contradicts (8.9). Therefore, the first 100 digits of $\sqrt{x}$ are all 9s, the same as for $x$ .

<u>A simpler proof:</u> We know that $0.999 < 0.99\ldots9 < 1$, so

$$\sqrt{0.999} < \sqrt{0.99...9} < \sqrt{1},$$

i.e. $$0.9994998... < \sqrt{0.99...9} < 1$$

Hence the first three significant digit of $\sqrt{0.999...9}$ with a hundred 9's is 999.

Problem #9:

Given any 50 integers, prove that you can choose some of them so that their sum is divisible by 50.

Solution:

Let's name the given integers $a_1, a_2, a_3, ..., a_{50}$. Consider the following 50 sums:

$$S_1 = a_1$$

$$S_2 = a_1 + a_2$$

$$S_3 = a_1 + a_2 + a_3$$

......

$$S_{50} = a_1 + a_2 + a_3 + ... + a_{50}$$

If one of $S_1, S_2, S_3, ..., S_{50}$ is divisible by 50, we are done.

Now assume none of $S_1, S_2, S_3, ..., S_{50}$ is divisible by 50, then each of $S_1, S_2, S_3, ..., S_{50}$ has a non-zero remainder when divided by 50. But there are only 49 non-zero remainders: 1,2,3,…,49. By the Dirichlet Principle, there must be two sums $S_k$ and $S_t$ (assume $k < t$) among $S_1, S_2, S_3, ..., S_{50}$ having the same remainder $r$. This means

$S_k = 50k_1 + r$ and $S_t = 50k_2 + r$

So,

$$S_t - S_k = (50k_2 + r) - (50k_1 + r) = 50k_2 - 50k_1 = 50(k_2 - k_1)$$

is divisible by 50. And $S_t - S_k = a_{k+1} + a_{k+2} + \ldots + a_t$ is a sum of some of $a_1, a_2, a_3, \ldots, a_{50}$.

## Helpful Notes:

## <The Dirichlet Principle>

Suppose we have at least n+1 people living in n houses. The Dirichlet principle says that in this case, there should be a house with at least 2 people in it.

## <The Remainder Theorem>

For any positive integers $a \geq b$, we can find unique integers k and r such that $a = kb + r$, where $0 \leq r < b$ .eg, when dividing 205 by 3, we will have 68 as the quotient and 1 as the remainder, that means

$205 = 68 \times 3 + 1$.